Mathematics II

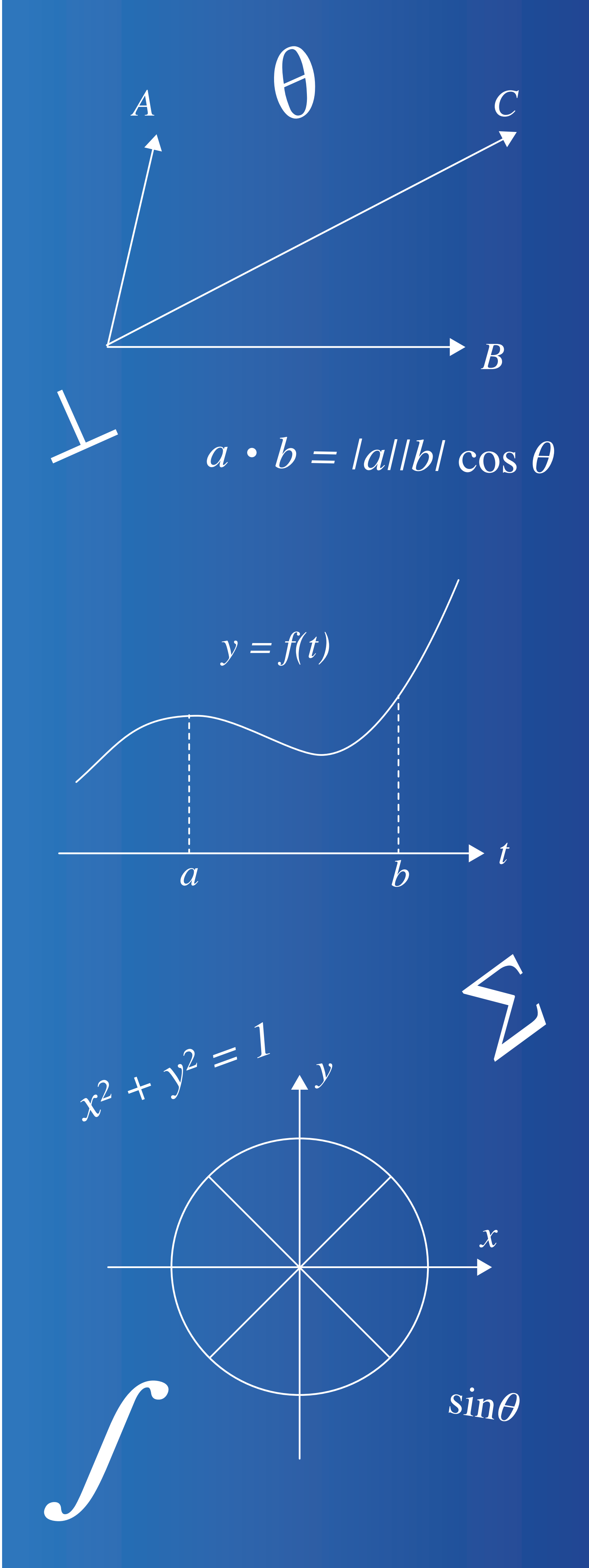
Egyptian Baccalaureate

2nd Year

Part Two

11

SPRIX



Mathematics

Egyptian Baccalaureate

2nd Year

Advised by
INTERNATIONAL BACCALAUREATE®

The International Baccalaureate™

has reviewed the pedagogical and assessment approaches underpinning this textbook.

Ministry of Education and Technical Education

New Administrative Capital

Cairo, Egypt

Introduction

In alignment with the vision of the Egyptian Baccalaureate System and its commitment to preparing learners for the demands of the twenty-first century, this Mathematics textbook for Grade 11 has been developed to provide students with a rigorous, engaging, and intellectually stimulating learning experience. The curriculum is designed to foster deep conceptual understanding, mathematical literacy, and the ability to apply mathematical knowledge and skills to real-world contexts.

The philosophy underpinning this textbook is grounded in inquiry-based learning, where students actively construct understanding through observation, questioning, investigation, and evidence-based reasoning. Learning begins with meaningful problems, authentic situations, and exploratory experiences that stimulate curiosity and encourage learners to seek explanations for the mathematical patterns, relationships, and structures that occur in the world around them. Through this approach, students are guided to discover patterns, formulate conjectures and explanations, and develop mathematical understanding before engaging with formal concepts, principles, procedures, and models.

The organization of each lesson reflects a coherent progression from exploration to conceptualization and application through the Point-Warm-Up-Explanation-Try-Exercise approach. Students are first introduced to carefully selected contexts, problems, and examples that connect new learning to prior knowledge and everyday experiences. These engaging introductions serve as a catalyst for inquiry, encouraging learners to analyze information, interpret representations, identify patterns, and construct initial explanations. Building upon this exploratory phase, students then engage with the core mathematical concepts, principles, procedures, and models that underpin these situations and provide a structured framework for understanding. The content is intentionally organized to promote deep conceptual understanding rather than rote memorization, enabling learners to recognize relationships among concepts, make meaningful connections between ideas, and apply their knowledge confidently in both familiar and unfamiliar contexts. Through the subsequent Try and Exercise stages, students are provided with opportunities to practice, apply, and assess their understanding, ultimately developing a robust, coherent, and interconnected understanding of mathematics.

This textbook provides extensive opportunities for problem solving, mathematical modeling, and independent investigation. Students are encouraged to test ideas, collect and analyze data, evaluate reasoning, and draw justified conclusions. Through these experiences, learners develop confidence in applying mathematical thinking and become active participants in the construction of knowledge rather than passive recipients of information.

Assessment is integrated throughout the learning process as a tool for both learning and improvement. A variety of formative and summative assessment opportunities are embedded within the lessons to help students monitor their progress, identify areas for growth, and demonstrate their understanding. Assessment tasks are carefully designed to reflect the nature of questions students may encounter in examinations while simultaneously promoting higher-order thinking, analytical reasoning, and the application of knowledge in unfamiliar contexts.

Particular emphasis has been placed on highlighting the Nature of Mathematics as a fundamental component of mathematical literacy. Students are provided with opportunities to understand how mathematical knowledge is developed, justified, refined, and communicated within mathematical communities. The curriculum also encourages appreciation of the contributions of mathematicians and innovators whose ideas and discoveries have shaped our understanding of the mathematical world and contributed to human progress.

The textbook systematically develops mathematical process skills, including problem solving, reasoning, representation, communication, modeling, estimation, pattern recognition, data interpretation, and proof. These skills are integrated within learning experiences to ensure that students not only acquire mathematical knowledge but also develop the competencies required to think and work mathematically.

In keeping with the aspirations of contemporary education, the curriculum promotes global awareness and international-mindedness by connecting mathematical concepts to scientific practices, technological innovations, environmental challenges, and sustainable development goals. Students are encouraged to appreciate the interconnected nature of knowledge and to recognize the role of mathematics in addressing issues that affect communities at local, national, and global levels.

Furthermore, the curriculum seeks to cultivate critical and creative thinking by engaging students in the analysis of information, evaluation of alternative strategies, generation of innovative solutions, and application of mathematical knowledge to authentic challenges. Learners are encouraged to approach problems as mathematicians, scientists, and engineers, employing systematic reasoning and design thinking to investigate questions and develop practical solutions.

A key feature of this textbook is the deliberate integration of mathematics with students' lives and experiences. Throughout the chapters, mathematical ideas are presented within meaningful contexts and supported by contemporary applications drawn from science, engineering, technology, economics, data, sustainability, and everyday life. Such connections reinforce the relevance of mathematics and help learners appreciate its significance in understanding and improving the world around them.

Equally important is the curriculum's commitment to fostering reflective learning. Students are encouraged to engage in self-assessment and metacognitive reflection, enabling them to monitor their understanding, evaluate their learning strategies, and take increasing responsibility for their own academic growth. By reflecting on both what they learn and how they learn, students develop the habits of lifelong learners capable of adapting to new challenges and opportunities.

Ultimately, this textbook aims to develop mathematically literate, responsible, and innovative learners who are equipped with the knowledge, skills, values, and dispositions necessary to thrive in a rapidly changing world. It is our hope that this learning journey will inspire students to explore mathematics with curiosity, think with rigor, act with responsibility, and contribute meaningfully to the advancement of society and the sustainable future of humanity.

MATHEMATICS

Textbook For Grade XI (Part 2)

Chapter 5 Trigonometric Functions

Chapter 6 Exponential and Logarithmic Functions

Chapter 7 Calculus: Differentiation and Integration

Chapter 8 Statistical Inference

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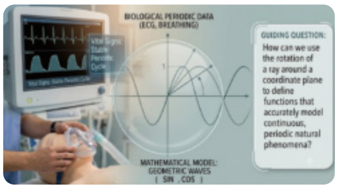
MATHEMATICS

Angles in Standard Position

Chapter 5 Trigonometric Functions

Concept 1 · Defining Angles and Basic Trigonometric Functions



When a medical professional monitors a patient's heart using an electrocardiogram (ECG), the screen displays a repeating, rhythmic wave. This visual representation of electrical activity keeps track of vital life signs. Similarly, respiratory therapists track how air pressure cycles in and out of human lungs during breathing.

These physiological patterns repeat over fixed intervals of time, mimicking the behavior of geometric waves. The mathematical tool used to model, analyze, and predict these cyclical biological systems is trigonometry. Guiding Question: How can the rotation of a ray in the coordinate plane be used to define trigonometric functions that model continuous and periodic phenomena in nature with accuracy?

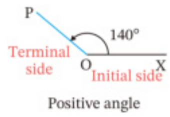
Learning Outcomes

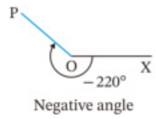
Identify the angle in standard position.

Point!

❗ On a plane, the half-line OP rotating around point O is called the **terminal side** (or terminal ray), and its initial position OX is called **the initial side**.

❗ The direction of rotation for the terminal side is **counter-clockwise** when the angle is positive, and **clockwise** when the angle is negative.





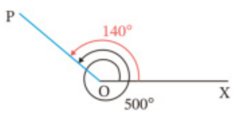
❗ Let α be the measure of an angles formed by the terminal side OP and the initial side OX. The angle θ expressed as

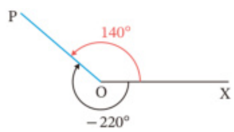
$\theta = \alpha + 360° \times n$ (where n is an integer)

is called **the general angle** represented by the terminal side OP.

Example: The terminal side OP for 140° coincides with the terminal side for 500° and − 220°.

$500° = 140° + 360° \times 1$ $\qquad -220° = 140° + 360° \times (-1)$





In this case, the general angle θ represented by terminal side OP can be expressed as

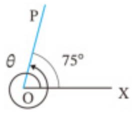
θ= **$140° + 360° \times n$ (where n is an integer)** .

Warm Up

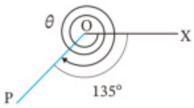
Solve the following problems.

1 Find the measure of angle θ in the figures below.

(a)



(b)



2 Draw the terminal side for the following angles.

(a) $300°$ **(b)** $-450°$

3 Express the following angles as general angles in the form $\alpha + 360° \times n$, where n is an integer and $0° \leq \alpha < 360°$.

(a) $420°$ **(b)** $-210°$

Explanation

1 **(a)** Since it represents one full counter-clockwise revolution plus 75°:

$360° \times 1 + 75° = 435°$

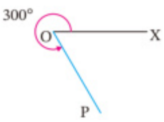
Therefore, <u>435°</u>

(b) Since it represents two full clockwise revolutions and a further 135°:

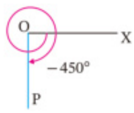
$360° \times (-2) - 135° = -855°$

Therefore, <u>−855°</u>

2 **(a)**

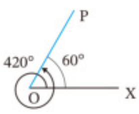


(b)



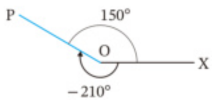
3 When expressing angles in the form $\alpha + 360° \times n$ (where n is an integer), it is helpful to sketch the terminal side.

(a)



From the figure above, <u>$60° + 360° \times n$ (where n is an integer)</u>.

(b)

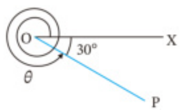


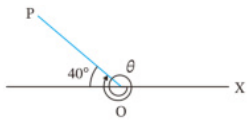
From the figure above, <u>$150° + 360° \times n$ (where n is an integer)</u>.

Try

Solve the following problems.

1 Find the measure of angle θ in the figures below.

(a) 

(b) 

2 Draw the terminal side for the following angles.

(a) $500°$ **(b)** $-120°$

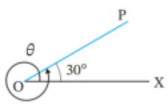
3 Express the following angles as general angles in the form $\alpha + 360° \times n$, where n is an integer and $0° \leq \alpha < 360°$.

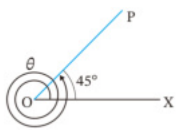
(a) $520°$ **(b)** $-140°$

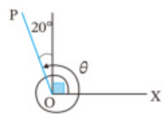
Exercise

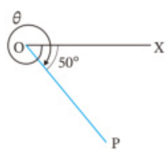
Solve the following problems.

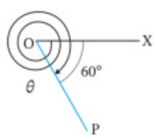
1 Find the measure of angle θ in the figures below.

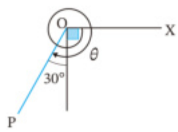
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

2 Draw the terminal side for the following angles.

(a) $470°$ **(b)** $780°$ **(c)** $225°$

(d) $-380°$ **(e)** $-990°$ **(f)** $-200°$

3 Express the following angles as general angles in the form $\alpha+360°\times n$, where n is an integer and $0° \leq \alpha < 360°$.

(a) $610°$ **(b)** $920°$ **(c)** $840°$

(d) $-430°$ **(e)** $-960°$ **(f)** $-20°$

★ Challenge

Express the following angles as general angles in the form $\alpha + 360° \times n$, where n is an integer, $0° \leq \alpha < 360°$: a) $129720°$ b) $-2592100°$

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

5-2

MATHEMATICS

Radian Measure

Chapter 5 Trigonometric Functions

Learning Outcomes

Convert angle measurements accurately between the sexagesimal system (degrees) and the circular system (radians).

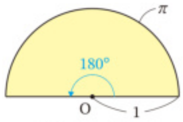
Point!

! Radian Measure

The method of expressing a central angle by the length of the arc is called radian measure.

When the arc length of a sector with radius 1 and central angle θ is l, $\theta = l$ radians.

Note that in radian measure, the unit “radian” is omitted.



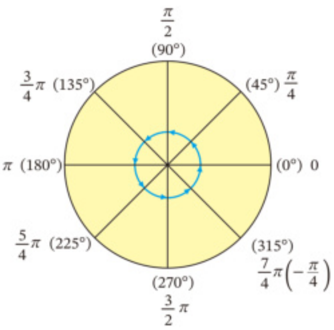
180° $= \pi$ radians

! To convert from degrees to radians, multiply the degrees by $\frac{\pi}{180}$

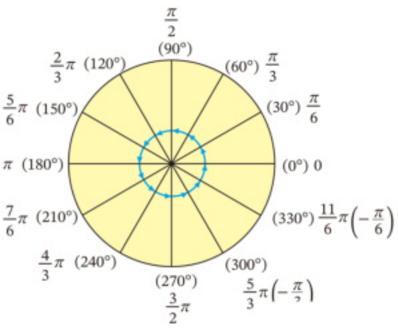
Example: Expressing 60° in radian measure: $60 \times \frac{\pi}{180} = \frac{\pi}{3}$

! Be able to sketch the terminal sides for the typical angles below in radians.

Dividing π(180°) into 4 parts,



Dividing π into 6 parts,



! In radian measure, the arc length of a sector is

(Radius) × (Central Angle)

Area is $\textbf{(Radius)}^2 \times \pi \times \dfrac{\textbf{(Central Angle)}}{2\pi}$

Multiply the circumference (or area) of a full circle by $\dfrac{\text{(Central Angle)}}{2\pi}$

Warm Up

Solve the following problems.

1 Convert 630° to radians.

2 Convert $-\frac{7}{4}\pi$ to degrees.

3 For a sector with radius 3 and central angle $\frac{4}{3}\pi$, find the following.

(a) Arc length l **(b)** Area S

Explanation

1 $630 \times \dfrac{\pi}{180} = \dfrac{7}{2}\pi$

Multiply degrees by $\frac{\pi}{180}$

2 Since $\pi = 180°$:

$-\dfrac{7}{4} \times 180° = -315°$

3 (a) $L = r \times \theta^{rad}$

$L = 3 \times \dfrac{4}{3}\pi = 4\pi$

(b) $S = r^2 \times \pi \times \dfrac{\frac{4}{3}\pi}{2\pi} = 3^2 \times \pi \times \dfrac{\frac{4}{3}\pi}{2\pi}$

$= 6\pi$

Multiply numerator and denominator by 3

Try

Solve the following problems.

1 Use the diagrams to express degrees in radian measure and complete the following table.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Radians																	

2 Convert the following angles to radians.

(a) 288° **(b)** −400°

3 Convert the following angles to degrees.

(a) $\dfrac{11}{6}\pi$ **(b)** $-\dfrac{3}{5}\pi$

4 For a sector with radius 6 and central angle $\dfrac{3}{4}\pi$, find the following.

(a) Arc length l **(b)** Area S

Exercise

Solve the following problems.

1 Convert the following angles to radians.

(a) 108° **(b)** 36° **(c)** −240° **(d)** −225°

(e) 420° **(f)** 400° **(g)** −570° **(h)** −2700°

2 Convert the following angles to degrees.

(a) $\dfrac{7}{6}\pi$ **(b)** $\dfrac{4}{5}\pi$ **(c)** $\dfrac{2}{3}\pi$ **(d)** $\dfrac{4}{3}\pi$

(e) $-\dfrac{5}{4}\pi$ **(f)** $-\dfrac{\pi}{2}$ **(g)** $-\dfrac{11}{2}\pi$ **(h)** $-\dfrac{7}{18}\pi$

3 Find the arc length l and area S for the following sectors.

(a) Sector with radius 6, central angle $\dfrac{7}{6}\pi$

(b) Sector with radius 4, central angle $\dfrac{2}{5}\pi$

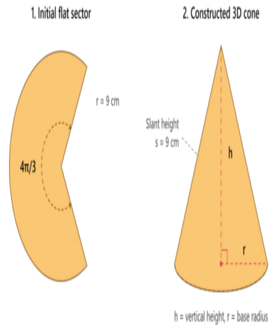
(c) Sector with radius 6, central angle $\dfrac{5}{4}\pi$

4 Draw the terminal side for the following angles.

(a) $\dfrac{\pi}{4}$ **(b)** $\dfrac{\pi}{3}$ **(c)** $\dfrac{5}{6}\pi$ **(d)** $\dfrac{3}{4}\pi$

(e) $\dfrac{4}{3}\pi$ **(f)** $\dfrac{7}{6}\pi$ **(g)** $-\dfrac{\pi}{3}$ **(h)** $-\dfrac{\pi}{4}$

★ Challenge



A sector with a radius of 9 cm and a central angle of 4π/3 radians is cut out from a flat sheet of paper. The two straight edges of this sector are joined together without overlapping to construct the lateral surface of a three-dimensional right circular cone. Determine the exact vertical height and the total volume of the resulting cone.

Work with a partner

Collaborate with your group members to answer questions 3 and 4, then discuss the answers

5-3

MATHEMATICS

Values of Trigonometric Functions

Chapter 5 Trigonometric Functions

Point!

! From now on, angles will be expressed in radians.

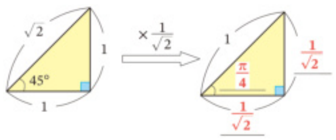
! Definition of Trigonometric Functions

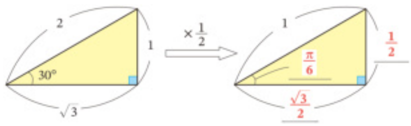
A circle with radius 1 centered at the origin is called **a unit circle.**

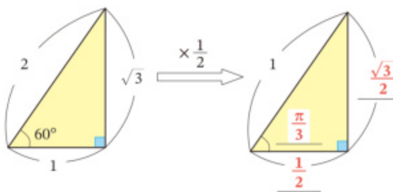
Let $\mathrm{P}(x,\ y)$ be the point of intersection of the terminal side of angle θ and the unit circle:

$\sin\theta = y$ (Sine) $\cos\theta = x$ (Cosine) $\tan\theta = \dfrac{y}{x}$ (Tangent)

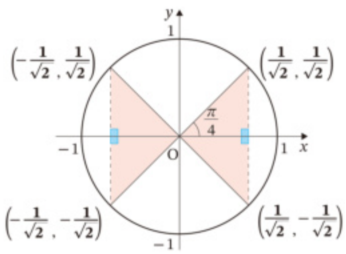
! To find the values of trigonometric functions, use right triangles with hypotenuse 1 as shown below.

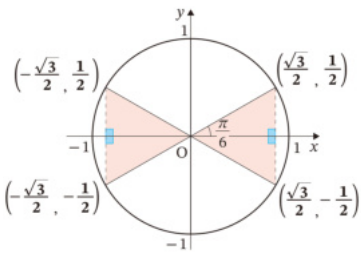


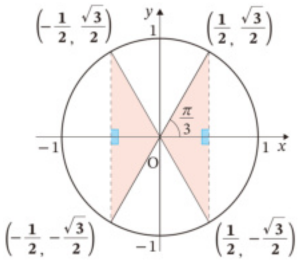




By applying the above right triangles to the unit circle, you can find the coordinates.



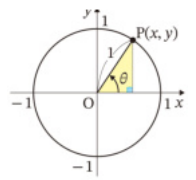




It is customary not to rationalize the denominators for $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{3}}$

Learning Outcomes

Derive the values of the basic trigonometric functions and their reciprocal functions for special and related angles using the unit circle.



If $\theta = \dfrac{\pi}{2},\ \dfrac{3}{2}\pi$, etc., $\tan\theta$ is undefined.

Warm Up

If $\theta = \frac{4}{3}\pi$, find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

Explanation

Always sketch a unit circle to help you solve.

Draw the terminal side OP for $\theta = \frac{4}{3}\pi$.

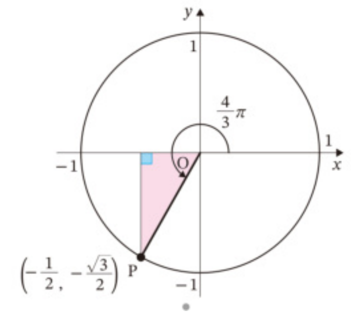
Drop a perpendicular from P to the x—axis to form a right triangle.

From the figure, $\sin\frac{4}{3}\pi = -\frac{\sqrt{3}}{2}$

$$\cos\frac{4}{3}\pi = -\frac{1}{2}$$

$$\tan\frac{4}{3}\pi = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$= \sqrt{3}$$



Y—coordinate

X—coordinate

$\tan\theta = \frac{y}{x}$

Try

Find the values of the following trigonometric functions. If the value is undefined, write "undefined".

(a) $\sin\frac{7}{4}\pi$

(b) $\cos\frac{\pi}{3}$

(c) $\tan\frac{\pi}{6}$

(d) $\sin\left(-\frac{2}{3}\pi\right)$

(e) $\cos(-\pi)$

(f) $\tan\left(-\frac{\pi}{2}\right)$

Exercise

Find the values of the following trigonometric functions. If the value is undefined, write "undefined".

(a) $\sin\frac{\pi}{3}$ **(b)** $\sin\frac{7}{6}\pi$ **(c)** $\sin\pi$

(d) $\cos\frac{5}{4}\pi$ **(e)** $\cos\frac{5}{3}\pi$ **(f)** $\cos\frac{7}{6}\pi$

(g) $\tan\frac{2}{3}\pi$ **(h)** $\tan\frac{5}{4}\pi$ **(i)** $\tan\pi$

(j) $\sin(-\frac{\pi}{6})$ **(k)** $\sin(-\frac{5}{4}\pi)$ **(l)** $\sin(-\frac{5}{3}\pi)$

(m) $\cos(-\frac{\pi}{4})$ **(n)** $\cos(-\frac{5}{6}\pi)$ **(o)** $\cos(-\frac{4}{3}\pi)$

(p) $\tan(-\frac{4}{3}\pi)$ **(q)** $\tan(-\frac{11}{6}\pi)$ **(r)** $\tan(-\frac{3}{2}\pi)$

★ Challenge

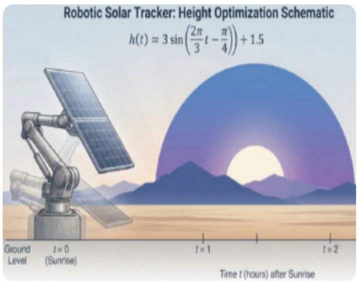
A robotic arm is programmed to track the movement of a solar panel.

The vertical height h(in meters) of the panel's tip relative to its base is modeled by the function:

$$h(t) = 3\sin\left(\frac{2\pi}{3}t - \frac{\pi}{4}\right) + 1.5$$

where (t)represents the time in hours after sunrise $(t = 0)$

Find the exact height of the panel's tip at $(t = 0)$hours. Do not use a calculator; leave your answer in exact radical form.



Work with a partner

Collaborate with your group members to answer parts d, e, and f, then discuss the answers.

5-4

MATHEMATICS

Finding Angles from Trigonometric Values

Chapter 5 Trigonometric Functions

Point!

! Method for Finding Angles from Trigonometric Values.

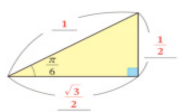
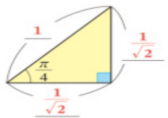
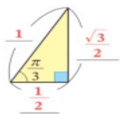
1. Draw a unit circle.

2. Draw the following lines based on the trigonometric function and find the points of intersections with the unit circle.

If $\sin\theta = a$, then $y = a$
If $\cos\theta = a$, then $x = a$
If $\tan\theta = a$, then $y = ax$

3. Find the angle θ in standard position whose terminal side passes through the intersection point.

To find θ, use the reference right triangles learned in Section $5-3$

Learning Outcomes

Calculate the measure of the angles from trigonometric values.

Warm Up

Solve the following equations for θ, where $0 \le \theta < 2\pi$

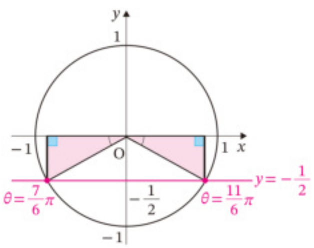
(a) $\sin\theta = -\dfrac{1}{2}$

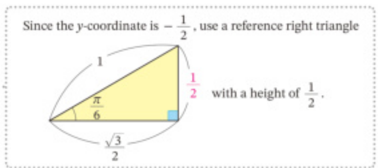
(b) $\cos\theta = \dfrac{1}{\sqrt{2}}$

(c) $\tan\theta = -\sqrt{3}$

Explanation

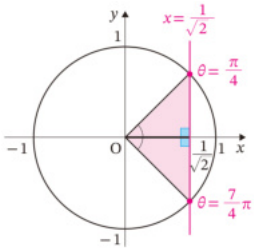
(a) Draw a unit circle and draw the line $y = -\dfrac{1}{2}$

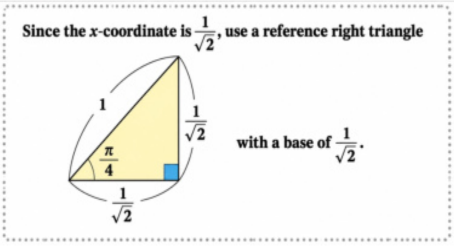




From the figure above, $\theta = \dfrac{7}{6}\pi$ or $\dfrac{11}{6}\pi$

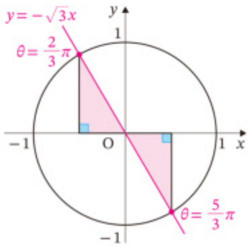
(b) Draw a unit circle and draw the line $x = \dfrac{1}{\sqrt{2}}$

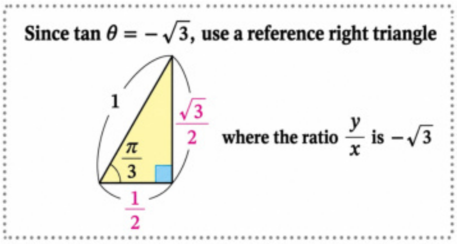




From the figure above, $\theta = \dfrac{\pi}{4}$ or $\dfrac{7}{4}\pi$

(c) Draw a unit circle and draw the line $y = -\sqrt{3}x$





From the figure above, $\theta = \dfrac{2}{3}\pi$ or $\dfrac{5}{3}\pi$

Try

Solve the following equations for θ, where $0 \le \theta < 2\pi$

(a) $\sin\theta = \dfrac{1}{\sqrt{2}}$ **(b)** $\cos\theta = -\dfrac{\sqrt{3}}{2}$ **(c)** $\tan\theta = \dfrac{1}{\sqrt{3}}$

(d) $\sin\theta = 0$ **(e)** $\cos\theta = \dfrac{1}{2}$ **(f)** $\tan\theta = -1$

Exercise

Solve the following equations for θ, where $0 \le \theta < 2\pi$

(a) $\sin\theta = \dfrac{1}{2}$ **(b)** $\sin\theta = \dfrac{\sqrt{3}}{2}$ **(c)** $\cos\theta = 0$

(d) $\cos\theta = 1$ **(e)** $\tan\theta = 0$ **(f)** $\tan\theta = \sqrt{3}$

(g) $\sin\theta = 1$ **(h)** $\sin\theta = -\dfrac{1}{\sqrt{2}}$ **(i)** $\cos\theta = -\dfrac{1}{\sqrt{2}}$

(j) $\cos\theta = -\dfrac{1}{2}$ **(k)** $\tan\theta = -\dfrac{1}{\sqrt{3}}$ **(l)** $\tan\theta = 1$

(m) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ **(n)** $\sin\theta = -1$ **(o)** $\cos\theta = -1$

(p) $\cos\theta = \dfrac{\sqrt{3}}{2}$

★ Challenge

Solve the following equation for θ, where $0 \le \theta < 360°$:
$\tan^2\theta + (1 - \sqrt{3})\tan\theta = \sqrt{3}$

Work with a partner

Collaborate with your group members to answer parts d, e, and f, then discuss the answers.

Reflect

How does moving from a fixed, right-triangle view of geometry to a rotational, unit-circle view change your understanding of what an "angle" can be?

5-5

MATHEMATICS

Fundamental Trigonometric Identities

Chapter 5 Trigonometric Functions

Concept 2 · Trigonometric Identities and Expression Evaluation

In the field of audio engineering and acoustic science, clear sound relies entirely on managing complex waveforms. When multiple instruments play simultaneously, or when a microphone picks up background static, the resulting sound wave is a messy, tangled combination of different frequencies. If software tried to process every single raw wave individually, the computational demand would crash the system or cause massive audio lag.

To solve this, noise-canceling headphones and audio compression algorithms use mathematical equivalence. They take sprawling, complicated wave equations and compress them into sleek, single-term expressions. By rewriting complex data into simpler, identical mathematical forms, technology filters out static and delivers crisp audio in real time. Guiding Question: How can foundational algebraic relationships between trigonometric ratios be used to systematically simplify complex, multi-term trigonometric expressions into a single, computable value?

Learning Outcomes

Apply fundamental trigonometric identities to simplify and verify trigonometric expressions.

Point!

! Signs of Trigonometric Functions

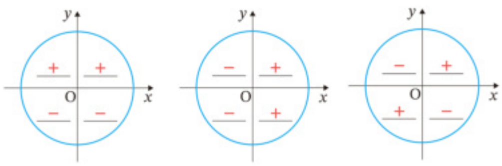
Let P be the point of intersection of the terminal side of angle θ and the unit circle:

$\sin\theta \rightarrow$ ***y*-coordinate** of P

$\cos\theta \rightarrow$ ***x*-coordinate** of P

$\tan\theta \rightarrow$ **Slope of OP**

Signs



! Fundamental Trigonometric Identities.

$$\sin^2\theta + \cos^2\theta = 1,\quad \tan\theta = \frac{\sin\theta}{\cos\theta},\quad 1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

! Finding $\cos\theta$ and $\tan\theta$ given $\sin\theta$. (The same approach applies when given $\cos\theta$)

1. Find $\cos\theta$ from $\sin^2\theta + \cos^2\theta = 1$ (or $\sin\theta$ if starting from $\cos\theta$).
2. Find $\tan\theta$ from $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

! How to find $\sin\theta$ and $\cos\theta$ from $\tan\theta$

1. Find $\cos\theta$ from $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$
2. Find $\sin\theta$ from $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

! To find $\sin\theta$ or $\cos\theta$ from $\sin^2\theta$ or $\cos^2\theta$, determine the sign by paying attention to the quadrant of the angle.

Refer to the diagram above for the signs of trigonometric functions.

Warm Up

Given one trigonometric value and the quadrant of angle θ, find the values of the other two trigonometric functions.

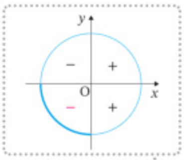
(a) $\sin\theta = -\frac{1}{5}$ (where θ is in Quadrant III)

(b) $\tan\theta = -\sqrt{6}$ $\left(\text{where } \frac{3}{2}\pi < \theta < 2\pi\right)$

Explanation

(a) From $\sin^2\theta + \cos^2\theta = 1$:

$$\cos^2\theta = 1 - \sin^2\theta$$
$$= 1 - (-\frac{1}{5})^2$$
$$= \frac{24}{25}$$



Since θ is in the 3rd quadrant, $\cos\theta <$ 0, so:

$$\cos\theta = -\frac{2\sqrt{6}}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{-\frac{1}{5}}{-2\frac{\sqrt{6}}{5}}$$
$$= \frac{1}{2\sqrt{6}}$$
$$= \frac{\sqrt{6}}{12}$$

In summary,

$$\underline{\cos\theta = -\frac{2\sqrt{6}}{5} \text{ or } \tan\theta = \frac{\sqrt{6}}{12}}$$

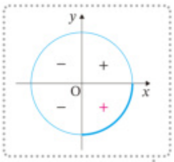
(b) From $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$:

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$
$$= 1 + (-\sqrt{6})^2$$
$$= 7$$

Solving for $\cos^2\theta$,

we get $\cos^2\theta = \frac{1}{7}$.

Since $\cos\theta > 0$ for $\frac{3}{2}\pi < \theta < 2\pi$:



$$\cos\theta = \frac{\sqrt{7}}{7}$$

From $\tan\theta = \frac{\sin\theta}{\cos\theta}$,

$$\sin\theta = \tan\theta \times \cos\theta$$
$$= -\sqrt{6} \times \frac{\sqrt{7}}{7}$$
$$= -\frac{\sqrt{42}}{7}$$

In summary,

$$\underline{\sin\theta = -\frac{\sqrt{42}}{7} \text{ or } \cos\theta = \frac{\sqrt{7}}{7}}$$

Multiply both the numerator and denominator by -5

Try

Given the angle θ and one of the values of $\sin\theta$, $\cos\theta$, or $\tan\theta$ as follows, find the other two values.

(a) $\cos\theta = -\frac{1}{\sqrt{3}}$ (where θ is in Quadrant III)

(b) $\tan\theta = -2$ (where θ is in Quadrant IV)

(c) $\sin\theta = \frac{3}{5}$ (where $\frac{\pi}{2} < \theta < \pi$)

(d) $\tan\theta = \frac{\sqrt{3}}{2}$ (where $\pi < \theta < \frac{3}{2}\pi$)

Exercise

Given the angle θ and one of the values of $\sin\theta$, $\cos\theta$, or $\tan\theta$ as follows, find the other two values.

(a) $\sin\theta = -\frac{3}{4}$ (where θ is in Quadrant III)

(b) $\cos\theta = \frac{2}{\sqrt{5}}$ (where θ is in Quadrant IV)

(c) $\tan\theta = -\sqrt{7}$ (where θ is in Quadrant IV)

(d) $\sin\theta = \frac{2}{3}$ (where θ is in Quadrant II)

(e) $\cos\theta = \frac{1}{4}$ (where θ is in Quadrant IV)

(f) $\tan\theta = -2\sqrt{2}$ (where θ is in Quadrant IV)

(g) $\tan\theta = -\frac{1}{3}$ (where $\frac{\pi}{2} < \theta < \pi$)

(h) $\sin\theta = \frac{2}{\sqrt{5}}$ (where $\frac{\pi}{2} < \theta < \pi$)

(i) $\cos\theta = \frac{5}{13}$ (where $\frac{3}{2}\pi < \theta < 2\pi$)

(j) $\sin\theta = -\frac{5}{13}$ (where $\frac{3}{2}\pi < \theta < 2\pi$)

(k) $\cos\theta = -\frac{4}{5}$ (where $\pi < \theta < \frac{3}{2}\pi$)

(l) $\tan\theta = -\frac{1}{2}$ (where $\frac{3}{2}\pi < \theta < 2\pi$)

★ Challenge

Using trigonometric identities, find the other two values, given that: $\tan\theta + \cot\theta \neq 25\ \ 12$ where $\pi < \theta < 2\pi$.

Work with a partner

Collaborate with your group members to answer parts c and d, then discuss the answers.

5-6
MATHEMATICS

Evaluating Trigonometric Expressions

Chapter 5 Trigonometric Functions

Point!

- Use $\sin^2\theta + \cos^2\theta = 1$ for evaluating trigonometric expressions.
- Use the following formula to evaluate $\sin^3\theta + \cos^3\theta$.

  $\sin^3\theta + \cos^3\theta = \underline{(\sin\theta + \cos\theta)\left(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right)}$
- To find $\sin\theta \pm \cos\theta$, determine the sign based on the quadrant of the angle.

Learning Outcomes

Apply fundamental trigonometric identities (Pythagorean identities, co-function identities, and even/odd function properties) to simplify and verify trigonometric expressions.

Warm Up

Given $\sin\theta + \cos\theta = \frac{1}{2}$, evaluate the following expressions, where θ is in Quadrant II.

(a) $\sin\theta\cos\theta$ **(b)** $\sin^3\theta + \cos^3\theta$ **(c)** $\sin\theta - \cos\theta$

$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

Explanation

(a) $\sin\theta + \cos\theta = \frac{1}{2}$

$(\sin\theta + \cos\theta)^2 = \frac{1}{4}$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{4}$

$1 + 2\sin\theta\cos\theta = \frac{1}{4}$

$\sin\theta\cos\theta = -\frac{3}{8}$

Square both sides and use $\sin^2\theta + \cos^2\theta = 1$

(b) $\sin^3\theta + \cos^3\theta$

$= (\sin\theta + \cos\theta)\left(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right)$

$= \frac{1}{2} \times (1 + \frac{3}{8})$

$= \frac{11}{16}$

- $\sin^2\theta + \cos^2\theta = 1$
- From **(a)**, $\sin\theta\cos\theta = -\frac{3}{8}$

(c) $(\sin\theta - \cos\theta)^2$

$= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$= 1 - 2 \times (-\frac{3}{8})$

$= \frac{7}{4}$

Square it and use $\sin^2\theta + \cos^2\theta = 1$

Since θ is in Quadrant II, $\sin\theta > 0$ and $\cos\theta < 0$, so:

$\sin\theta - \cos\theta > 0$

Therefore, $(\sin\theta - \cos\theta)^2 = \frac{7}{4}$

$\underline{\sin\theta - \cos\theta = \frac{\sqrt{7}}{2}}$

Try

Given $\sin\theta + \cos\theta = \frac{1}{3}$, evaluate the following expressions, where θ is in Quadrant II.

(a) $\sin\theta\cos\theta$ **(b)** $\sin^3\theta + \cos^3\theta$ **(c)** $\sin\theta - \cos\theta$

Exercise

Solve the following problems.

1 Given $\sin\theta - \cos\theta = \frac{\sqrt{3}}{2}$, evaluate the following expressions, where θ is in Quadrant I.

(a) $\sin\theta\cos\theta$ **(b)** $\sin^3\theta - \cos^3\theta$ **(c)** $\sin\theta + \cos\theta$

2 Given $\sin\theta + \cos\theta = \frac{\sqrt{3}}{2}$, evaluate the following expressions, where θ is in Quadrant IV.

(a) $\sin\theta\cos\theta$ **(b)** $\sin^3\theta + \cos^3\theta$ **(c)** $\sin\theta - \cos\theta$

3 Given $\sin\theta - \cos\theta = \frac{1}{2}$, evaluate the following expressions, where θ is in Quadrant III.

(a) $\sin\theta\cos\theta$ **(b)** $\sin^3\theta - \cos^3\theta$ **(c)** $\sin\theta + \cos\theta$

★ Challenge

Given that $\sin\theta - \cos\theta = 0.5$ and θ is in Quadrant III, evaluate the exact value of $\sin^6\theta - \cos^6\theta$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-7

MATHEMATICS

Proving Identities Involving Trigonometric Functions

Chapter 5 Trigonometric Functions

Point!

! In proving identities involving trigonometric functions, show that LHS = RHS.

! Frequently Used Transformations.

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = \underline{1}$
- $1 - \sin^2\theta = \underline{\cos^2\theta}$
- $1 - \cos^2\theta = \underline{\sin^2\theta}$

! To add or subtract fractions, **combine the fractions using a common denominator**.

Learning Outcomes

Apply fundamental trigonometric identities (Pythagorean identities, co-function identities, and even/odd function properties) to simplify and verify trigonometric expressions.

Warm Up

Prove that $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1}{\tan\theta} = \dfrac{1}{\sin\theta}$

Explanation

[Proof]

$$\text{LHS} = \frac{\sin\theta}{1+\cos\theta} + \frac{1}{\dfrac{\sin\theta}{\cos\theta}}$$

Rewrite $\tan\theta$ as $\dfrac{\sin\theta}{\cos\theta}$, multiply numerator and denominator by cos θ.

$$= \frac{\sin\theta}{1+\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos\theta(1+\cos\theta)}{(1+\cos\theta)\sin\theta}$$

Combine the fractions by finding a common denominator.

$$= \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{(1+\cos\theta)\sin\theta}$$

$\sin^2\theta + \cos^2\theta = 1$

$$= \frac{1+\cos\theta}{(1+\cos\theta)\sin\theta}$$

$$= \frac{1}{\sin\theta}$$

$$= \text{RHS}$$

Therefore, $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1}{\tan\theta} = \dfrac{1}{\sin\theta}$

Try

Prove the following identities.

(a) $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

(b) $\dfrac{\cos\theta}{1-\sin\theta}-\tan\theta=\dfrac{1}{\cos\theta}$

Exercise

Prove the following identities.

(a) $\tan^2\theta-\sin^2\theta=\tan^2\theta\sin^2\theta$

(b) $\tan\theta+\dfrac{1}{\tan\theta}=\dfrac{1}{\sin\theta\cos\theta}$

(c) $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{1}{\cos\theta}=\tan\theta$

(d) $\dfrac{\tan\theta}{\sin\theta}-\dfrac{\sin\theta}{\tan\theta}=\sin\theta\tan\theta$

(e) $\dfrac{1+\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{2}{\cos\theta}$

(f) $\dfrac{\tan^2\theta-\sin^2\theta}{\sin^2\theta}=\tan^2\theta$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

★ Challenge

Prove that for all values of x where the expressions are defined, the following equation holds true:

$$\frac{1+\sin x-\cos x}{1+\sin x+\cos x}+\frac{1+\sin x+\cos x}{1+\sin x-\cos x}=2\csc x$$

Reflect

A student claims that because $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, the equation $\tan\theta\times\cos\theta=\sin\theta$ is an identity that holds true for absolutely all real numbers. Critically analyze this statement. Is there any specific condition or value of θ where this relationship completely breaks down? Explain your reasoning.

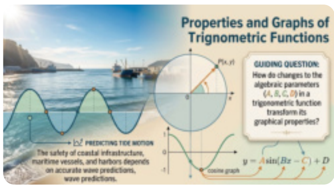
5-8

MATHEMATICS

Properties of Trigonometric Functions

Chapter 5 Trigonometric Functions

Concept 3 · Properties and Graphs of Trigonometric Functions



Oceanography and coastal safety depend entirely on predicting the motion of tides. Coastal areas experience a continuous rise and fall in water levels caused by the gravitational pull of the Moon and the Sun, combined with the rotation of the Earth.

If emergency planners or maritime vessels miscalculate the peak height of a tide or the timing of its low point, harbors can flood, and container ships can run aground. Because tidal changes repeat on a predictable schedule, scientists do not plot them as erratic lines, but as smooth, continuous geometric waves. Guiding Question: How do changes to the algebraic parameters in a trigonometric function mathematically transform its graphical properties, such as its vertical stretch, horizontal compression, and shifting?

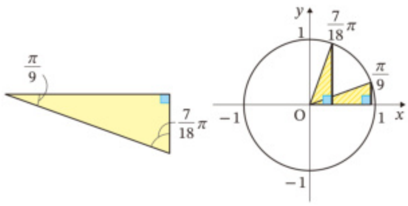
Learning Outcomes

Analyze the periodic properties of trigonometric functions (range, amplitude, period, and horizontal shift), represent them graphically, and interpret their behavior.

Point!

! For problems expressing trigonometric functions in terms of acute angles, draw the terminal side of the angle on a unit circle to form a right triangle, and consider it in Quadrant I.

Example:

Making this 

! Trigonometric functions of angles such as $\theta+\frac{\pi}{2}$, $\theta+\pi$, $\theta+2n\pi$ (where n is an integer), and $-\theta$ can be expressed as trigonometric functions of θ.

Consider two right triangles with the acute angle at the bottom left.

Warm Up

Solve the following problems.

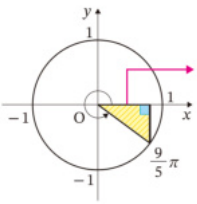
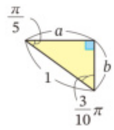
1 Express $\sin\frac{9\pi}{5}$ as a trigonometric function of an acute angle

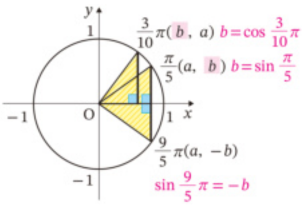
2 Simplify

$$\sin\theta+\sin\left(\theta+\frac{\pi}{2}\right)+\sin(\theta+\pi)+\sin\left(\theta+\frac{3\pi}{2}\right)$$

Explanation

1 Since $\frac{3}{2}\pi < \frac{9}{5}\pi < 2\pi$, $\frac{9}{5}\pi$ is an angle in the Quadrant IV.


Using this,  it looks like the figure below.



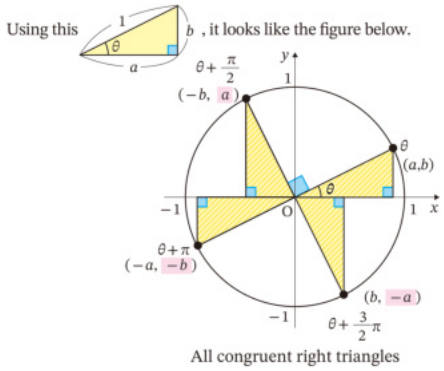
From the figure above, values equal to $\sin\frac{9}{5}\pi$ are $\underline{-\sin\frac{\pi}{5} \text{ and } -\cos\frac{3}{10}\pi}$

$2\pi - \frac{9}{5}\pi = \frac{\pi}{5}$

$\frac{\pi}{2} - \frac{\pi}{5} = \frac{3}{10}\pi$

2 Express trigonometric functions of different angles all in terms of trigonometric functions of the same angle(mainly θ).

Although the measure of θ is unknown, it is helpful to sketch it with an appropriate measure.



All congruent right triangles

From the figure on the left: $\sin(\theta + \frac{\pi}{2}) = a = \cos\theta$

$\sin(\theta + \pi) = -b = -\sin\theta$

$\sin\left(\theta + \frac{3}{2}\pi\right) = -a = -\cos\theta$

Therefore, the value of the required expression is:

$\sin\theta + \cos\theta - \sin\theta - \cos\theta = \underline{0}$

The answer will always include one sin and one cos.

Try

Solve the following problems.

1 Express $\cos\left(-\frac{\pi}{6}\right)$ as a trigonometric function of an acute angle.

2 Express $\sin\frac{8}{7}\pi$ as a trigonometric function of an acute angle.

3 Simplify $\cos\left(\frac{\pi}{2} - \theta\right)\sin(\pi - \theta) - \sin\left(\frac{3}{2}\pi + \theta\right)\cos(\pi - \theta)$

Exercise

Solve the following problems.

1 Express the following trigonometric functions as trigonometric functions of acute angles.

(a) $\cos\dfrac{17}{5}\pi$ **(b)** $\cos\dfrac{5}{7}\pi$

(c) $\sin\left(-\dfrac{7}{8}\pi\right)$ **(d)** $\sin\left(-\dfrac{19}{8}\pi\right)$

2 Find the following values.

(a) $\tan\left(-\dfrac{\pi}{6}\right)$ **(b)** $\tan\left(-\dfrac{\pi}{4}\right)$

3 Simplify the following expressions.

(a) $\sin\theta+\cos(\pi+\theta)-\cos(\pi-\theta)+\sin(\pi+\theta)$

(b) $\sin\left(\theta+\dfrac{\pi}{2}\right)\cos(\theta+\pi)+\sin(-\theta)\cos\left(\dfrac{\pi}{2}-\theta\right)$

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

★ Challenge

Simplify:

$$\sin\left(\frac{3\pi}{2}-\theta\right)\sec(\pi+\theta)-\tan\left(\theta-\frac{\pi}{2}\right)\cot\left(\frac{5\pi}{2}+\theta\right)$$

5-9

MATHEMATICS

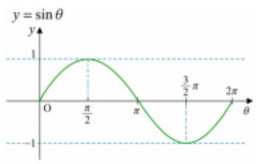
Graphs of Trigonometric Functions (1)

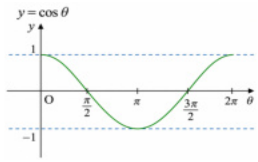
Chapter 5 Trigonometric Functions

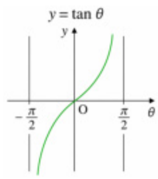
Point!

! Graphs of $y=\sin\theta,\ y=\cos\theta,\ y=\tan\theta$.

- Memorize a single cycle of the following and be able to draw them.







Period: 2π
Range: $-1\leq y\leq 1$

Period: 2π
Range: $-1\leq y\leq 1$

Period: π
Range: All real numbers

- Mark the scale on the θ–axis from $-\pi$ to 2π in increments of $\frac{\pi}{2}$

Learning Outcomes

Analyze the periodic properties of trigonometric functions (range, amplitude, period, and horizontal shift), represent them graphically, and interpret their behavior.

$\theta=-\frac{\pi}{2},\ \frac{\pi}{2}$ are asymptotes that the graph approaches indefinitely.

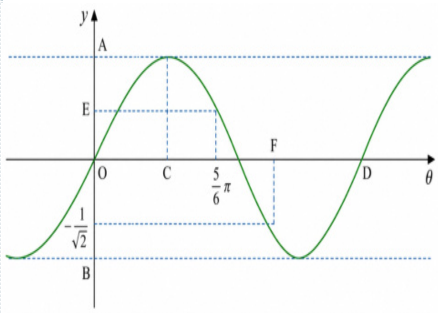
Warm Up

Solve the following problems.

1 Draw the following graphs. Also, find their periods.

(a) $y=\sin\theta$ **(b)** $y=\cos\theta$ **(c)** $y=\tan\theta$

2 The figure on the right is the graph of $y=\sin\theta$
Determine the values of the scales A to F in the figure.

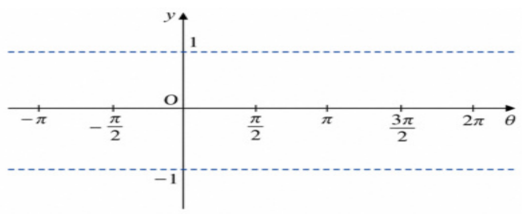


Explanation

1 **(a)**

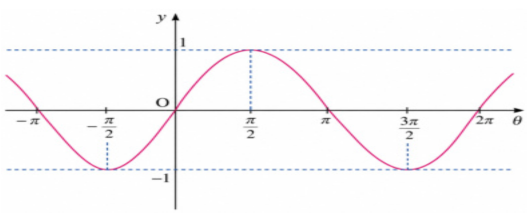
❶ First draw the θ–axis and y–axis, and mark the θ–axis in increments of $\frac{\pi}{2}$ from $-\pi$ to 2π.

❷ Draw horizontal dashed lines at $y=\pm1$



❸ Sketch the standard graph for one period.

❹ Extend the graph periodically over the remaining domain.



Period is 2π

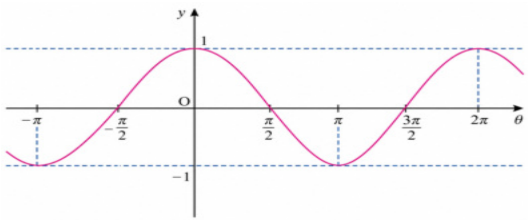
(b)

❶ First draw the θ–axis and y–axis, and mark the θ–axis in increments of $\dfrac{\pi}{2}$ from $-\pi$ to 2π.

❷ Draw horizontal dashed lines at $y = \pm 1$

❸ Sketch the standard graph for one period.

❹ Extend the graph periodically over the remaining domain.



Period is 2π

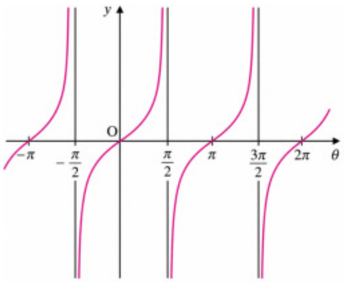
(c)

❶ First draw the θ–axis and y–axis, and mark the θ–axis in increments of $\dfrac{\pi}{2}$ from $-\pi$ to 2π.

❷ Draw asymptotes at $\theta = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3}{2}\pi$.

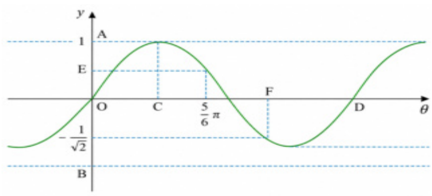
❸ Sketch the standard graph for one period.

❹ Extend the graph periodically over the remaining domain.



Period is π

2 As when drawing a graph, mark the scale in increments of $\dfrac{\pi}{2}$ on the θ–axis, and draw $y = \pm 1$ on the graph.



From this, A is 1, B is − 1, C is $\dfrac{\pi}{2}$, D is 2π.

Since E is the value of $\sin\dfrac{5}{6}\pi$, it is $\dfrac{1}{2}$

F is the point in the range $\pi \leq \theta \leq \dfrac{3}{2}\pi$ where $\sin\theta = -\dfrac{1}{\sqrt{2}}$ from the graph, so $\dfrac{5}{4}\pi$.

In summary, A: 1, B: − 1, C: $\dfrac{\pi}{2}$, D: 2π, E: $\dfrac{1}{2}$, F: $\dfrac{5}{4}\pi$.

Try

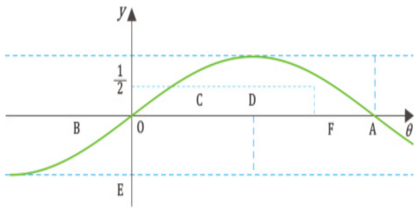
Solve the following problems.

1 Draw the following graphs. Also, find their periods.

(a) $y = \sin\theta$ **(b)** $y = \cos\theta$ **(c)** $y = \tan\theta$

2 The figure on the right is the graph of the function $y = \cos\theta$.

Determine the values of A through F on the axes.



Exercise

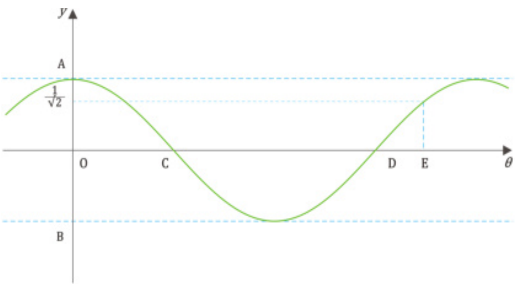
Solve the following problems.

1 Draw the following graphs. Also, find their periods.

(a) $y = \sin\theta$ **(b)** $y = \cos\theta$ **(c)** $y = \tan\theta$

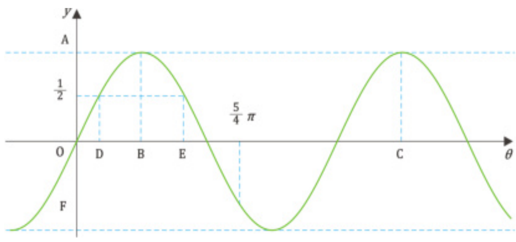
2 The figure on the right is the graph of the function $y = \cos\theta$

Determine the values of A through E on the axes.



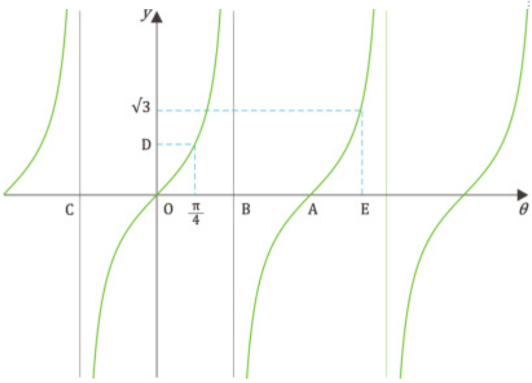
3 The figure on the right is the graph of the function $y = \sin\theta$.

Determine the values of A through F on the axes.



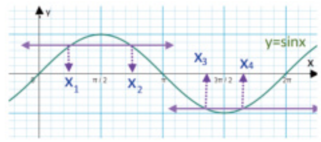
4 The figure on the right is the graph of the function $y = \tan\theta$.

Determine the values of A through E on the axes.



★ Challenge

The opposite graph represents: $y = sinx,\ 0 < x < 2\pi$



Then find: $x_1 + x_2 + x_3 + x_4$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

5-10
MATHEMATICS

Graphs of Trigonometric Functions (2)

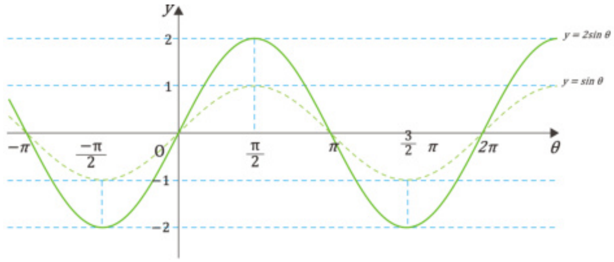
Chapter 5 Trigonometric Functions

Point!

! Graph of $y = a\sin\theta$.

The graph of $y = \sin\theta$ **stretched vertically by a factor of a** (Same applies to $y = a\cos\theta, y = a\tan\theta$.)

Example: Graph of $y = 2\sin\theta$.



Period is 2π.

Range is

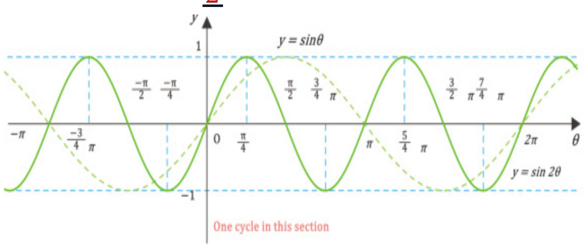
$-2 \leq y \leq 2$

! Graph of $y = \sin b\theta$

The graph of $y = \sin\theta$ **compressed horizontally by a factor of $\frac{1}{b}$**. (Same applies to $y = \cos b\theta, y = \tan b\theta$)

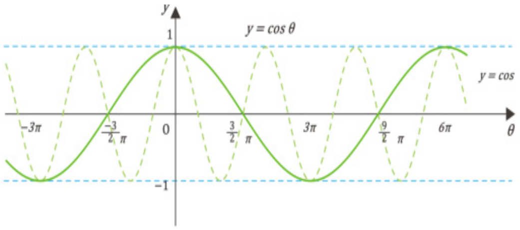
→ The values on the θ–axis and the period are multiplied by $\frac{1}{b}$.

Example: Graph of $y = \sin 2\theta$: The graph of $y = \sin\theta$ compressed horizontally by $\frac{1}{2}$



- Mark the scale on the θ–axis in increments of $\frac{\pi}{4}$
- Period is π
- Range is $-1 \leq y \leq 1$

Example: Graph of $y = \cos\frac{\theta}{3}$: The graph of $y = \cos\theta$ stretched horizontally by 3



- Mark the scale on the θ–axis in increments of $\frac{3}{2}\pi$
- Period is 6π
- Range is $-1 \leq y \leq 1$

Learning Outcomes

Analyze the periodic properties of trigonometric functions (range, amplitude, period, and horizontal shift), represent them graphically, and interpret their behavior.

Multiply by the reciprocal of b.

Warm Up

Draw the graphs of the following functions. Also, find their periods.

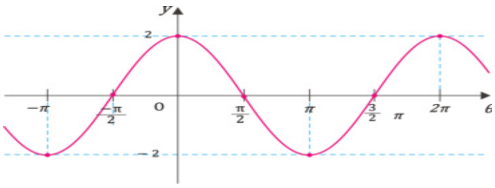
(a) $y = 2\cos\theta$

(b) $y = \tan\dfrac{\theta}{2}$

Explanation

1 The graph of $y = 2\cos\theta$ is:

The graph of $y = \cos\theta$ stretched vertically by a factor of 2



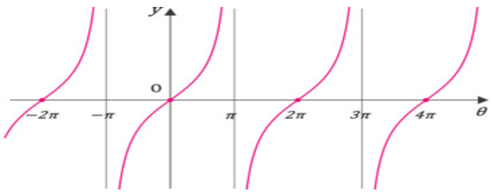
Mark the θ–axis in increments of $\dfrac{\pi}{2}$
Mark ± 2 on the y-axis.

Period is 2π.

2 The graph of $y = \tan\dfrac{\theta}{2}$ is the graph of $y = \tan\theta$ stretched horizontally by a factor of 2

Since the coefficient of θ is 1, the period does not change.

Multiply by the reciprocal of $\dfrac{1}{2}$



Mark the θ–axis in increments of π

Period is 2π.

It becomes twice the period π of $y = \tan\theta$

Try

Draw the graphs of the following functions. Also, find their periods.

(a) $y = 2\sin\theta$

(b) $y = \tan 2\theta$

(c) $y = \cos\dfrac{\theta}{2}$

Exercise

Draw the graphs of the following functions. Also, find their periods.

(a) $y = 3\sin\ \theta$

(b) $y = 3\cos\ \theta$

(c) $y = \frac{1}{2}\cos\ \theta$

(d) $y = \sin 3\ \theta$

(e) $y = \cos 2\theta$

(f) $y = \tan 3\theta$

(g) $y = \sin\frac{\theta}{2}$

(h) $y = \cos\ \frac{\theta}{3}$

(i) $y = \tan\ \frac{\theta}{3}$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

★ Challenge

Draw the graph of $y = -2|\sin(3x)|$.
Also, find its period.

5-11 Graphs of Trigonometric Functions (3)

MATHEMATICS

Chapter 5 Trigonometric Functions

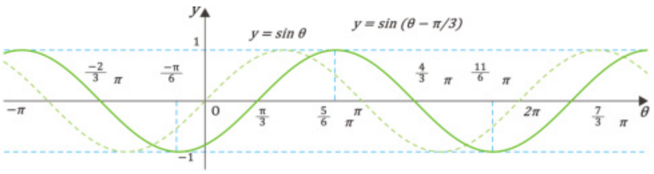
Point!

Graph of $y = \sin(\theta - p)$.

The graph of $y = \sin\theta$ **shifted horizontally by p**. (Same applies to $y = \cos(\theta - p)$, $y = \tan(\theta - p)$.)

→ Mark the θ–axis in increments of $\frac{\pi}{2}$ starting from **p**.

Example: Graph of $y = \sin\left(\theta - \frac{\pi}{3}\right)$ Mark the θ–axis sin increments of $\frac{\pi}{2}$ from $\frac{\pi}{3}$



Period is 2π

Range is

$-1 \leq y \leq 1$

Learning Outcomes

Analyze the periodic properties of trigonometric functions (range, amplitude, period, and horizontal shift), represent them graphically, and interpret their behavior.

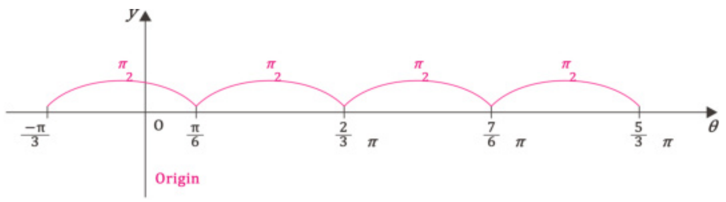
Warm Up

Draw the graph of the function $y = \sin\left(\theta - \frac{\pi}{6}\right)$. Also, find its period.

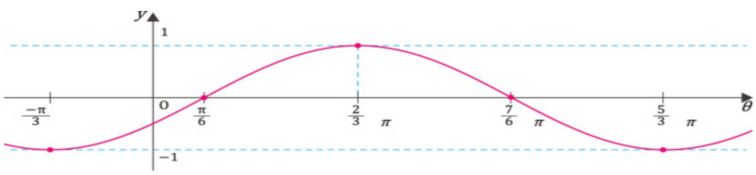
Explanation

$y = \sin(\theta - \frac{\pi}{6})$ is the graph of $y = \sin\theta$ shifted horizontally by $\frac{\pi}{6}$.

Therefore, mark the θ–axis in increments of $\frac{\pi}{2}$ starting from $\frac{\pi}{6}$.



Sketch the standard graph for one period starting from $\frac{\pi}{6}$, and extend the graph periodically over the remaining domain.



Period is 2π.

If you are taught at school to mark the y–intercept, be sure to draw it.

Substitute $\theta = 0$ into

$y = \sin\left(\theta - \frac{\pi}{6}\right)$:

$y = \sin(-\frac{\pi}{6}) = -\frac{1}{2}$

Mark 3 scales to the right and 1 scale to the left starting from $\frac{\pi}{6}$.

Try

Draw the graphs of the following functions. Also, find their periods.

1 $y = \sin(\theta - \frac{\pi}{3})$

2 $y = \cos\left(\theta + \frac{\pi}{4}\right)$

3 $y = \tan\left(\theta - \frac{\pi}{2}\right)$

Since the coefficient of θ is 1, the period does not change.

Exercise

Solve the following problems.

1 Draw the graphs of the following functions. Also, find their periods.

(a) $y = \sin\left(\theta + \frac{\pi}{4}\right)$ **(b)** $y = \sin\left(\theta - \frac{\pi}{2}\right)$

(c) $y = \sin\left(\theta + \frac{\pi}{3}\right)$

2 Draw the graphs of the following functions. Also, find their periods.

(a) $y = \cos\left(\theta - \frac{\pi}{3}\right)$ **(b)** $y = \cos\left(\theta - \frac{\pi}{4}\right)$

(c) $y = \cos\left(\theta + \frac{\pi}{6}\right)$

3 Draw the graphs of the following functions. Also, find their periods.

(a) $y = \tan\left(\theta + \frac{\pi}{6}\right)$ **(b)** $y = \tan\left(\theta - \frac{\pi}{3}\right)$

(c) $y = \tan\left(\theta - \frac{\pi}{4}\right)$

★ Challenge

Draw the graph of $y = 3\cos|x + \frac{\pi}{3}|$. Also, find its period.

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

5-12
MATHEMATICS

Graphs of Trigonometric Functions (4)

Chapter 5 Trigonometric Functions

Point!

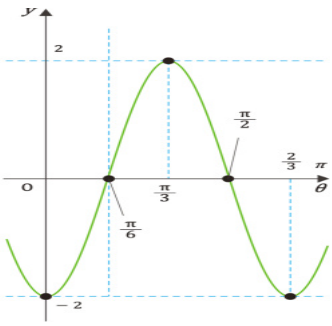
! For the graph of $y = a\sin(b\theta - q)$, factor out b from the angle and transform it to $y = a\sin b(\theta - \frac{q}{b})$

Apply the transformations in the following order: shift horizontally by $\frac{q}{b}$, scale horizontally by a factor of $\frac{1}{b}$, and scale vertically by a factor of a.

Example: Graph of $y = 2\sin(3\theta - \frac{\pi}{2})$

Rewrite as $y =$ $2\sin 3\left(\theta - \frac{\pi}{6}\right)$

→ Shift horizontally by $\frac{\pi}{6}$, scale horizontally by a factor of $\frac{1}{3}$, and scale vertically by a factor of 2



Period is $\frac{2}{3}\pi$

Range is $-2 \leq y \leq 2$

Learning Outcomes

Analyze the periodic properties of trigonometric functions (range, amplitude, period, and horizontal shift), represent them graphically, and interpret their behavior.

Apply the transformations from the inside out.

Warm Up

Draw the graphs of the following functions. Also, find their periods.

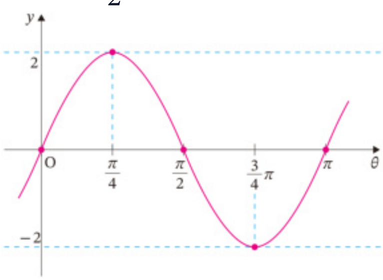
1 $y = 2\cos(2\theta - \frac{\pi}{2})$

2 $y = \tan\left(\frac{\theta}{2} - \frac{\pi}{6}\right)$

Explanation

1 Rewriting $y = 2\cos(2\theta - \frac{\pi}{2})$ gives $y = 2\cos 2\left(\theta - \frac{\pi}{4}\right)$

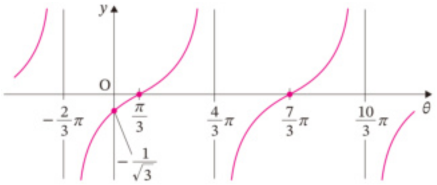
This is the graph of $y = \cos\theta$ shifted horizontally by $\frac{\pi}{4}$, scaled horizontally by a factor of $\frac{1}{2}$, and scaled vertically by a factor of 2



Period is π.

2 Rewriting $y = \tan(\frac{\theta}{2} - \frac{\pi}{6})$ gives $y = \tan\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)$

This is the graph of $y = \tan\theta$ shifted horizontally by $\frac{\pi}{3}$ and scaled horizontally by a factor of 2



Period is 2π.

Mark the y—intercept.

Try

Draw the graphs of the following functions. Also, find their periods.

(a) $y = 3\sin\frac{\theta}{2}$

(b) $y = \tan\left(2\theta - \frac{\pi}{2}\right)$

(c) $y = 2\cos\left(2\theta - \frac{\pi}{3}\right)$

Exercise

Draw the graphs of the following functions. Also, find their periods.

(a) $y = 2\cos\frac{\theta}{2}$

(b) $y = 3\sin 2\theta$

(c) $y = 2\cos 2\theta$

(d) $y = 2\cos\left(\theta - \frac{\pi}{4}\right)$

(e) $y = 2\sin\left(\theta - \frac{\pi}{3}\right)$

(f) $y = \tan(2\theta - \frac{\pi}{3})$

(g) $y = 3\cos(2\theta - \frac{\pi}{2})$

(h) $y = 3\sin(2\theta - \frac{2}{3}\pi)$

(i) $y = 2\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$

★ Challenge

Draw the graph of $y = 3sin|2x - \pi|$. Also, find its period.

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

Reflect

Consider the standard parent tangent function, y = tan x. Unlike sin and cos graphs, the tan graph has no defined amplitude and contains vertical breaks (asymptotes) at regular intervals. Using your understanding of the quotient identity tan x = sin x/cos x, explain mathematically why the graph of y = tan x explodes toward infinity at x = π/2 and x = 3π/2, and why it cannot possess an amplitude.

5-13

MATHEMATICS

Equations Involving Trigonometric Functions

Chapter 5 Trigonometric Functions

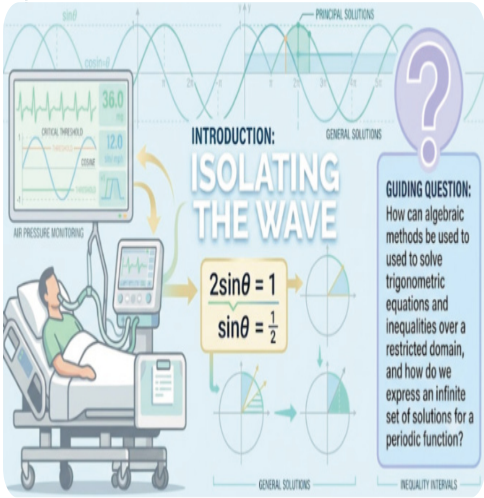
Concept 4 · Solving Equations and Inequalities Involving Trigonometric Functions

Medical technologies like patient ventilation systems rely heavily on precision monitoring. A mechanical ventilator delivers oxygenated air to a patient's lungs in a rhythmic cycle, regulating the fluid air pressure within the chest cavity. If the internal pressure drops too low, the lungs fail to absorb oxygen efficiently; if it remains elevated above a critical threshold for too long, delicate lung tissue sustains severe barotrauma.

To prevent injury, engineers and clinicians program medical alarms using wave functions. Finding the exact moments when a cyclical biological variable hits a critical threshold, or identifying the duration of time it spends outside a safe boundary, requires isolating specific values along a repeating wave. Guiding Question: How can algebraic methods be used to solve trigonometric equations and inequalities over a restricted domain, and how do we express an infinite set of solutions for a periodic function?

Learning Outcomes

Solve trigonometric equations.



Point!

❗ For equations involving trigonometric functions, rewrite the equation in the form $\sin\theta = a$ to find θ.(Same for $\cos\theta$, $\tan\theta$.)

❗ To find the general solution for θ, answer based on the period of the trigonometric function.

• The period of $\sin\theta$ and $\cos\theta$ is $2\pi \rightarrow$ Add $\underline{2n\pi}$ to the solution found in $0 \leq \theta < 2\pi$

• The period of $\tan\theta$ is $\pi \rightarrow$ Add $\underline{n\pi}$ to one of the solutions found in $0 \leq \theta < \pi$

See 5 – 4

Warm Up

Solve the following equations.

(a) $-2\cos\theta = \sqrt{2}$ (where $0 \leq \theta \leq 2\pi$)

(b) $\tan\theta + \sqrt{3} = 0$ (find the general solution)

Explanation

(a) $-2\cos\theta = \sqrt{2}$

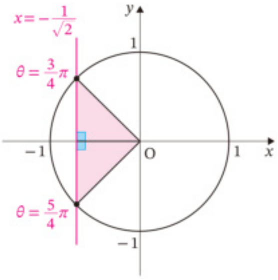
$$\cos\theta = -\frac{\sqrt{2}}{2}$$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

Rewrite in the form $\cos\theta = a$

Transform to $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

Sketching the unit circle

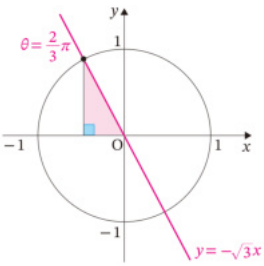


From the figure above: The solutions for $\cos\theta = -\frac{1}{\sqrt{2}}$ in the range $0 \leq \theta < 2\pi$ are:

$\theta = \frac{3}{4}\pi$ or $\frac{5}{4}\pi$

(b) $\tan\theta = -\sqrt{3}$

Sketching the unit circle in the range $0 \leq \theta < \pi$:



From the figure above, the angle where $\tan\theta = -\sqrt{3}$ is $\theta = \frac{2}{3}\pi$

Therefore, $\theta = \frac{2}{3}\pi + n\pi$ (where n is an integer)

Be sure to write (where n is an integer)

Common Mistakes

Therefore, $\theta = \frac{2}{3}\pi + 2n\pi$ (where n is an integer)

You overlooked that the period of $\tan\theta$ is π, unlike $\sin\theta$ and $\cos\theta$.

Try

Solve the following equations.

(a) $2\cos\theta - 1 = 0$ (where $0 \leq \theta \leq 2\pi$)

(b) $\tan\theta - 1 = 0$ (where $0 \leq \theta \leq 2\pi$)

(c) $2\sin\theta = \sqrt{2}$ (find the general solution)

Exercise

Solve the following equations.

(a) $2\sin\theta + 1 = 0$ (where $0 \le \theta \le 2\pi$)

(b) $2\sin\theta = \sqrt{3}$ (where $0 \le \theta \le 2\pi$)

(c) $3\sin\theta + 3 = 0$ (where $0 \le \theta \le 2\pi$)

(d) $\sqrt{2}\cos\theta + 1 = 0$ (where $0 \le \theta \le 2\pi$)

(e) $2\cos\theta = \sqrt{3}$ (where $0 \le \theta \le 2\pi$)

(f) $2\cos\theta - \sqrt{2} = 0$ (where $0 \le \theta \le 2\pi$)

(g) $\tan\theta + 1 = 0$ (where $0 \le \theta \le 2\pi$)

(h) $\sqrt{3}\tan\theta = 1$ (where $0 \le \theta \le 2\pi$)

(i) $\sqrt{3}\tan\theta = 3$ (where $0 \le \theta \le 2\pi$)

(j) $2\sin\theta + \sqrt{3} = 0$ (find the general solution)

(k) $\cos\theta + 1 = 0$ (find the general solution)

(l) $\sqrt{3}\tan\theta + 3 = 0$ (find the general solution)

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

★ Challenge

Solve the following equation:

$\tan^2\theta - (2 + \sqrt{3})\tan\theta + 2\sqrt{3} = 0$ *(Find the general solution)*

5-14

MATHEMATICS

Inequalities Involving Trigonometric Functions

Chapter 5 Trigonometric Functions

Learning Outcomes

Solve trigonometric inequalities.

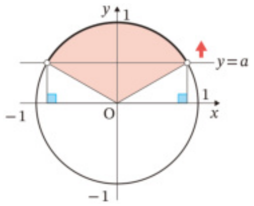
Point!

! How to Solve Inequalities Involving Trigonometric Functions.

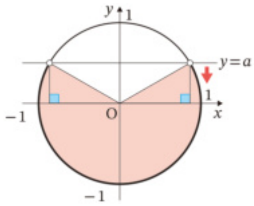
❶ Solve the equation obtained by replacing the inequality sign with an equal sign.

❷ Find the range of θ using the unit circle drawn when solving the equation.

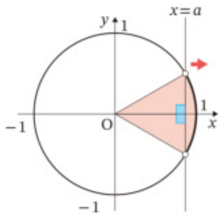
If $\sin\theta > a$, the arc **above** $y = a$



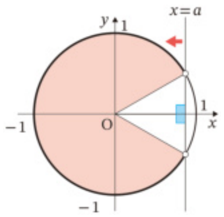
If $\sin\theta < a$, the arc **below** $y = a$



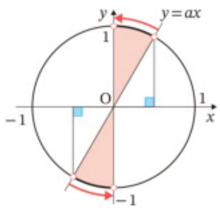
If $\cos\theta > a$, the arc to the **right** of $x = a$



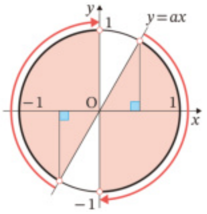
If $\cos\theta < a$, the arc to the **left** of $x = a$



If $\tan\theta > a$, the terminal side is rotated **counterclockwise** from the line $y = ax$.



If $\tan\theta < a$, the terminal side is rotated **clockwise** from the line $y = ax$.



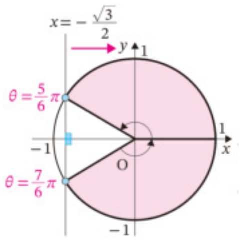
Warm Up

Solve the following inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $2\cos\theta > -\sqrt{3}$ **(b)** $\tan\theta \leq \dfrac{1}{\sqrt{3}}$

Explanation

(a)



$2\cos\theta > -\sqrt{3}$

$\cos\theta > -\dfrac{\sqrt{3}}{2}$

Rewrite the inequality in the form $\cos\theta > a$

The solutions for $\cos\theta = -\dfrac{\sqrt{3}}{2}$ in the range $0 \leq \theta < 2\pi$ are:

$\theta = \dfrac{5}{6}\pi$ or $\dfrac{7}{6}\pi$

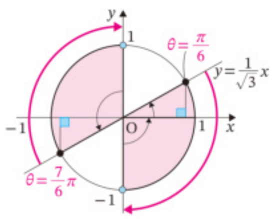
Start from $\theta = 0$

θ approaches 2π

From the figure on the left, the solution to the inequality is:

$\underline{0 \leq \theta < \dfrac{5}{6}\pi \text{ or } \dfrac{7}{6}\pi < \theta < 2\pi}$

(b)



$\tan\theta \leq \dfrac{1}{\sqrt{3}}$

The solutions for $\tan\theta = \dfrac{1}{\sqrt{3}}$ in the range $0 \leq \theta < 2\pi$ are:

$\theta = \dfrac{\pi}{6}$ or $\dfrac{7}{6}\pi$

From the figure on the left, the solution to the inequality is:

$\underline{0 \leq \theta \leq \dfrac{\pi}{6} \text{ or } \dfrac{\pi}{2} < \theta \leq \dfrac{7}{6}\pi \text{ or } \dfrac{3}{2}\pi < \theta < 2\pi}$

$\tan\dfrac{\pi}{2}$ and $\tan\dfrac{3}{2}\pi$ are undefined, so they are not included.

Try

Solve the following inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $2\sin\theta \leq -1$ **(b)** $\cos\theta > \dfrac{1}{2}$ **(c)** $\tan\theta > 1$

Exercise

Solve the following inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\sin\theta > \dfrac{\sqrt{3}}{2}$ **(b)** $\sin\theta - \dfrac{\sqrt{2}}{2} > 0$ **(c)** $2\sin\theta + \sqrt{3} \leq 0$

(d) $\cos\theta > 0$ **(e)** $\cos\theta - \dfrac{1}{2} \leq 0$ **(f)** $2\cos\theta + \sqrt{2} \leq 0$

(g) $\tan\theta \leq -\dfrac{1}{\sqrt{3}}$ **(h)** $\tan\theta + \sqrt{3} > 0$ **(i)** $\tan\theta + 1 \geq 0$

★ Challenge

Solve the following inequality for θ, where $0 \leq \theta < 2\pi$:

$2\sin^2\theta - \cos\theta - 1 > 0$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-15

MATHEMATICS

Equations and Inequalities Involving Trigonometric Functions (1)

Chapter 5 Trigonometric Functions

Point!

! For expressions containing both $\sin\theta$ and $\cos\theta$ like $2\cos^2\theta - 3\sin\theta = 0$, rewrite them as an expression in terms of a single trigonometric function using $\underline{\sin^2\theta + \cos^2\theta = 1}$

! Solve quadratic equations or inequalities involving $\sin\theta$ by treating $\sin\theta$ as a single variable.(Same for $\cos\theta$) • However, be careful that $\sin\theta$ must satisfy $\underline{-1 \leq \sin\theta \leq 1}$

Learning Outcomes

Solve trigonometric equations and inequalities (simple and compound), and determine their general solutions within specified intervals.

Warm Up

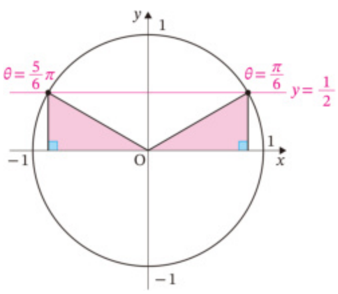
Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $2\cos^2\theta - 3\sin\theta = 0$

(b) $2\sin^2\theta - 3\cos\theta - 3 \leq 0$

Explanation

(a)



$2\cos^2\theta - 3\sin\theta = 0$

$2(1 - \sin^2\theta) - 3\sin\theta = 0$

Simplifying gives $2\sin^2\theta + 3\sin\theta - 2 = 0$

Solving this gives $\sin\theta = \frac{1}{2}$ or -2

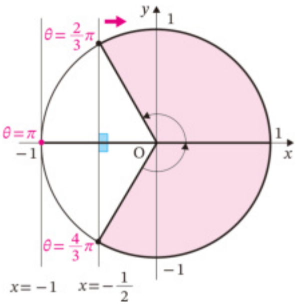
Since $-1 \leq \sin\theta \leq 1$, $\sin\theta = -2$ is not possible, so $\sin\theta = \frac{1}{2}$

From the figure on the left, solving this in the range $0 \leq \theta < 2\pi$ gives $\underline{\theta = \frac{\pi}{6} \text{ or } \frac{5}{6}\pi}$

Rewrite as an expression in terms of only $\sin\theta$

Solve this as you would the quadratic equation $2x^2 + 3x - 2 = 0$

(b)



$2\sin^2\theta - 3\cos\theta - 3 \leq 0$

$2(1-\cos^2\theta) - 3\cos\theta - 3 \leq 0$

Simplifying gives $2\cos^2\theta + 3\cos\theta + 1 \geq 0$

Solving this gives $\cos\theta \leq -1$ or $-\frac{1}{2} \leq \cos\theta$ ......(i)

Also, $-1 \leq \cos\theta \leq 1$ ......(ii)

From (i) and (ii), $\cos\theta = -1$ or $-\frac{1}{2} \leq \cos\theta \leq 1$

From the figure on the left, the required solution is:

$0 \leq \theta \leq \frac{2}{3}\pi$ or $\theta = \pi$ or $\frac{4}{3}\pi \leq \theta < 2\pi$

Rewrite as an expression in terms of only $\cos\theta$

Solve this as you would the quadratic inequality $2x^2 + 3x + 1 \geq 0$

Try

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $2\sin^2\theta - \sin\theta = 0$

(b) $2\cos^2\theta - \sin\theta = 2$

(c) $2\sin^2\theta - 4 < 5\cos\theta$

Exercise

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $2\sin^2\theta + \sin\theta - 1 = 0$

(b) $2\sin^2\theta + 3\cos\theta = 0$

(c) $2\sin^2\theta - \cos\theta = 2$

(d) $2\cos^2\theta - 5\cos\theta - 3 = 0$

(e) $2\cos^2\theta + \sin\theta - 1 = 0$

(f) $2\cos^2\theta + 3\sin\theta = 3$

(g) $2\cos^2\theta + 3\sin\theta \leq 0$

(h) $2\sin^2\theta + \cos\theta > 1$

(i) $2\cos^2\theta - 4 \leq 5\sin\theta$

★ Challenge

Solve the following inequality for θ, where $0 \leq \theta < 2\pi$:

$$\frac{1 - 2\sin^2\theta + \sin\theta}{\cos^2\theta} \geq 1$$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-16
MATHEMATICS

Equations and Inequalities Involving Trigonometric Functions (2)

Chapter 5 Trigonometric Functions

Point!

! If the expression in the argument is not simply θ, substitute t for the expression.

! If substituting with t, always find **the range of t**, and solve the equation/inequality within that range.

Learning Outcomes

Solve trigonometric equations and inequalities (simple and compound), and determine their general solutions within specified intervals.

Warm Up

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\sin\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{1}{2}$

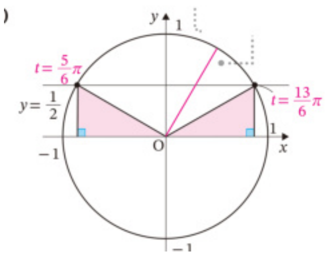
(b) $\sin\left(\theta + \dfrac{\pi}{6}\right) > -\dfrac{1}{2}$

Explanation

(a) Letting $\theta + \dfrac{\pi}{3} = t$, the range of t is:

$\dfrac{\pi}{3} \leq t < \dfrac{7}{3}\pi$...... (i)

$\sin t = \dfrac{1}{2}$ within the range of (i), from the figure on the left:



$t = \dfrac{5}{6}\pi$ or $\dfrac{13}{6}\pi$

Therefore, $\theta + \dfrac{\pi}{3} = \dfrac{5}{6}\pi,\ \dfrac{13}{6}\pi$

$\theta = \dfrac{\pi}{2}$ or $\dfrac{11}{6}\pi$

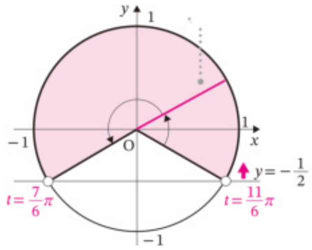
Since $\dfrac{\pi}{3} \leq t < \dfrac{7}{3}\pi$, consider one full revolution from here.

Add $\dfrac{\pi}{3}$ to each part of the inequality $0 \leq \theta < 2\pi$.

Now that the equation is solved for t, substitute the original expression back for t.

Subtract $\dfrac{\pi}{3}$ from each.

Since $\frac{\pi}{6} \leq t < \frac{13}{6}\pi$, consider one full revolution from here.



(b) Letting $\theta + \frac{\pi}{6} = t$, the range of t is:

$\frac{\pi}{6} \leq t < \frac{13}{6}\pi$...... (i)

Solving $\sin t > -\frac{1}{2}$ within the range of (i), from the figure on the left:

$\frac{\pi}{6} \leq t < \frac{7}{6}\pi$ or $\frac{11}{6}\pi < t < \frac{13}{6}\pi$

Therefore, $\frac{\pi}{6} \leq \theta + \frac{\pi}{6} < \frac{7}{6}\pi$ or $\frac{11}{6}\pi < \theta + \frac{\pi}{6} < \frac{13}{6}\pi$

$0 \leq \theta < \pi$ or $\frac{5}{3}\pi < \theta < 2\pi$

Try

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

(b) $\tan\left(\theta + \frac{\pi}{4}\right) = \sqrt{3}$

(c) $\cos\left(\theta + \frac{\pi}{4}\right) < -\frac{\sqrt{2}}{2}$

Exercise

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\cos\left(\theta + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

(b) $\cos\left(\theta + \frac{\pi}{6}\right) = -\frac{1}{2}$

(d) $\tan\left(\theta + \frac{\pi}{4}\right) = -1$

(c) $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$

(e) $\tan\left(\theta + \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$

(f) $\tan\left(\theta + \frac{\pi}{6}\right) = 1$

(g) $\sin\left(\theta + \frac{\pi}{6}\right) \leq \frac{\sqrt{3}}{2}$

(h) $\sin\left(\theta + \frac{\pi}{4}\right) \geq \frac{\sqrt{2}}{2}$

(i) $\cos\left(\theta + \frac{\pi}{3}\right) > \frac{1}{2}$

Reflect

A student is attempting to solve the equation $tan x \times cos\, x = \frac{1}{2}$ over the domain [0°, 360°[. They begin by dividing both sides of the equation by $cosx$,leaving them with $tanx = \frac{1}{2cosx}$ Evaluate this strategy. Did this first step make the equation easier or harder to solve? Propose an alternative algebraic substitution using fundamental identities that simplifies the expression without separating the interacting terms.

★ Challenge

Solve the following inequality for θ,where $0 \leq \theta < 2\pi$

$$4sin^2\left(\theta + \frac{\pi}{6}\right) \geq 1$$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-17

MATHEMATICS

Maximum and Minimum of Trigonometric Functions (1)

Chapter 5 Trigonometric Functions

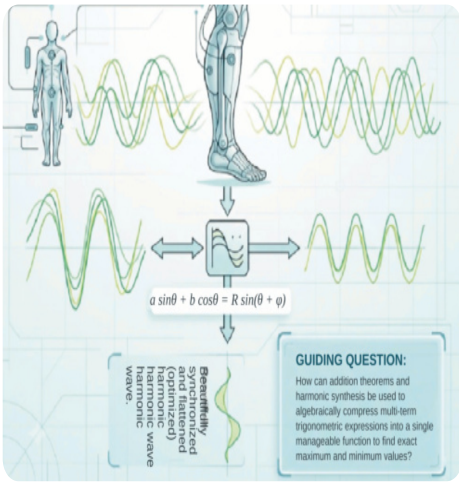
Concept 5 · Advanced Theorems and Applications

In electrical engineering and structural biomechanics, analyzing combined frequencies is essential. When alternating current (AC) electricity travels through a power grid, or when wireless medical devices transmit telemetry data from an implant, the resulting electronic signal is rarely a simple, isolated wave. Instead, multiple currents or biological rhythms intersect, creating composite waves.

If engineers were forced to compute every overlapping signal individually, communication systems would suffer massive latency and power losses. Advanced trigonometric identities allow us to take distinct, overlapping harmonic frequencies and synthesize them into a single, optimized wave function. By blending multiple variables mathematically, we can determine the exact peak thresholds and zero-points of intricate physical systems. Guiding Question: How can addition theorems and harmonic synthesis be used to algebraically compress multi-term trigonometric expressions into a single manageable function to find exact maximum and minimum values?

Learning Outcomes

Determine the maximum and minimum values of trigonometric functions over different intervals, and infer their behavior.



Point!

- To find the maximum and minimum of trigonometric functions, let $\sin\theta = t$ or $\cos\theta = t$, and use the graph of the function of t.
- If substituting with t, always write $-1 \leq t \leq 1$.

Warm Up

Find the maximum and minimum values of the function $y = \sin^2\theta + \cos\theta + 1$, where $0 \leq \theta < 2\pi$.

Also, find the corresponding values of θ.

Explanation

$y = \sin^2\theta + \cos\theta + 1$

$y = (1 - \cos^2\theta) + \cos\theta + 1$

$y = -\cos^2\theta + \cos\theta + 2$

Let $\cos\theta = t$, then $-1 \leq t \leq 1$

$y = -t^2 + t + 2$

$\quad = -\left(t - \frac{1}{2}\right)^2 + \frac{9}{4}$

Drawing this graph gives the figure on the right, so from the graph:

Maximum value of $\frac{9}{4}$ for $t = \frac{1}{2}$.

If $t = \frac{1}{2}$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ or $\frac{5}{3}\pi$.

Also, minimum value of 0 for $t = -1$

If $t = -1$, $\cos\theta = -1$, $\theta = \pi$.

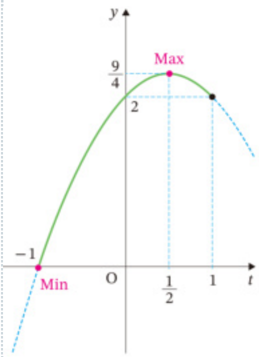
Therefore, max value of $\frac{9}{4}$ for $\theta = \frac{\pi}{3}$ or $\frac{5}{3}\pi$, min value of 0 for $\theta = \pi$.

Rewrite as an expression in terms of only $\cos\theta$.

Since $\cos\theta = t$, write $-1 \leq t \leq 1$

To find the maximum and minimum of a quadratic function, complete the square and sketch the graph.

Substitute $\cos\theta$ back for t.



Try

Find the maximum and minimum values of the function $y = \cos^2\theta - \sin\theta$, where $0 \leq \theta < 2\pi$. Also, find the corresponding values of θ.

Exercise

Find the maximum and minimum values of the following functions, where $0 \leq \theta < 2\pi$. Also, find the corresponding values of θ.

(a) $y = \sin^2\theta + \sin\theta - 1$

(b) $y = \cos^2\theta - \cos\theta + 1$

(c) $y = 2\sin\theta + \cos^2\theta - 1$

(d) $y = -2\sin^2\theta - 2\cos\theta + 1$

★ Challenge

Find the maximum and minimum values of the following function, where $0 \leq \theta < 2\pi$. Also, find the corresponding values of θ

$y = \sin^2\theta - 4\cos\theta + 2$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-18
MATHEMATICS

Addition Theorems

Chapter 5 Trigonometric Functions

Point!

! Addition Theorems.

$\sin(\alpha+\beta) =$ $\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\cos(\alpha+\beta) =$ $\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha-\beta) =$ $\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\cos(\alpha-\beta) =$ $\cos\alpha\cos\beta+\sin\alpha\sin\beta$

! Method for Calculating Trigonometric Values Using Addition Theorems.

• Convert from radians to **degrees** .

• Rewrite the angle in the form $45° \pm 30° \times n$.

! If the value of $\sin\theta$ or $\cos\theta$ is known, use $\sin^2\theta+\cos^2\theta=1$ to find the other.

Learning Outcomes

Apply the sum and difference formulas for two angles to calculate trigonometric values without using a calculator.

Warm Up

Solve the following problems.

1 Find the following values.

(a) $\sin 75°$ **(b)** $\cos\dfrac{\pi}{12}$

2 Given $\sin\alpha=\dfrac{\sqrt{3}}{2}$ and $\cos\beta=-\dfrac{3}{5}$, evaluate the following expressions, where α is in Quadrant II and β is in Quadrant III.

(a) $\sin(\alpha+\beta)$ **(b)** $\cos(\alpha-\beta)$

Explanation

1 **(a)** Since 75° = 45° + 30°:

$\sin(45°+30°)$

$=\sin45°\cos30°+\cos45°\sin30°$

$=\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}$

$=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$=\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Rewrite as $45° \pm 30° \times n$.

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

(b) Since $\frac{\pi}{12} = 15° = 45° - 30°$:

$\cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Convert to degrees.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

2 Since the addition theorem is used to find the values of $\sin(\alpha+\beta)$ and $\cos(\alpha-\beta)$, first find $\cos\alpha$ given $\sin\alpha$.

Since $\sin\alpha = \frac{\sqrt{3}}{2}$, from $\sin^2\alpha + \cos^2\alpha = 1$:

$\cos^2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{1}{4}$

Since α is in Quadrant II, $\cos\alpha = -\frac{1}{2}$.

Similarly, find $\sin\beta$ given $\cos\beta$.

Since $\cos\beta = -\frac{3}{5}$, from $\sin^2\beta + \cos^2\beta = 1$:

$\sin^2\beta = 1 - (-\frac{3}{5})^2$

$= \frac{16}{25}$

Since β is in Quadrant III, $\sin\beta = -\frac{4}{5}$.

Using these:

(a)

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{\sqrt{3}}{2}\times(-\frac{3}{5}) + (-\frac{1}{2})\times(-\frac{4}{5})$

$= \underline{\frac{4-3\sqrt{3}}{10}}$

(b)

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= (-\frac{1}{2})\times(-\frac{3}{5}) + \frac{\sqrt{3}}{2}\times(-\frac{4}{5})$

$= \underline{\frac{3-4\sqrt{3}}{10}}$

Try

Solve the following problems.

1 Find the following values.

(a) $\sin 15°$ **(b)** $\cos\frac{7}{12}\pi$

2 Given $\sin\alpha = \frac{3}{5}$ and $\cos\beta = -\frac{4}{5}$, evaluate the following expressions, where $0 < \alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$.

(a) $\sin(\alpha+\beta)$ **(b)** $\cos(\alpha+\beta)$

Exercise

Solve the following problems.

1 Write the expressions that apply to **(a)** to **(d)** below.

$\sin(\alpha+\beta)=$ **(a)**

$\sin(\alpha-\beta)=$ **(b)**

$\cos(\alpha+\beta)=$ **(c)**

$\cos(\alpha-\beta)=$ **(d)**

2 Find the following values.

(a) $\sin 105°$ **(b)** $\sin\frac{7}{12}\pi$ **(c)** $\sin\frac{13}{12}\pi$

(d) $\cos 165°$ **(e)** $\cos 15°$ **(f)** $\cos\frac{11}{12}\pi$

3 Given $\sin\alpha=\frac{3}{5}$ and $\cos\beta=-\frac{12}{13}$, evaluate the following expressions, where α is in Quadrant I and β is in Quadrant II.

(a) $\sin(\alpha-\beta)$ **(b)** $\cos(\alpha-\beta)$

4 Given $\sin\alpha=\frac{3}{5}$ and $\cos\beta=-\frac{2\sqrt{5}}{5}$, evaluate the following expressions, where $\frac{\pi}{2}<\alpha<\pi$ and $\frac{\pi}{2}<\beta<\pi$.

(a) $\sin(\alpha-\beta)$ **(b)** $\cos(\alpha+\beta)$

5 Given $\sin\alpha=\frac{5}{13}$ and $\cos\beta=-\frac{3}{5}$, evaluate the following expressions, where $0<\alpha<\frac{\pi}{2}$ and $\pi<\beta<\frac{3}{2}\pi$.

(a) $\sin(\alpha+\beta)$ **(b)** $\cos(\alpha+\beta)$

★ Challenge

Given $sin\alpha=0.6,\ cos\beta=0.8 and tan\gamma=0.75$,where $\alpha,\ \beta,\ and\gamma$ are three acute angles, evaluate without using a scientific calculator:

$sin(\alpha+\beta+\gamma)$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

5-19

MATHEMATICS

Angle Between Two Lines

Chapter 5 Trigonometric Functions

Point!

! Addition Theorem for Tangent.

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $\quad\quad$ $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

! Let α and β be the angles that two lines $y = m_1x + k_1$ and $y = m_2x + k_2$ make with the positive x—axis, respectively.

- $\tan\alpha = m_1$, $\tan\beta = m_2$

The angle θ between the two lines can be expressed as $\theta = \underline{\alpha - \beta}$.

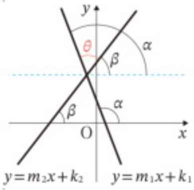
→ Calculate $\tan\theta$ to find θ.

! In problems finding the angle between two lines in the range $0 < \theta < \dfrac{\pi}{2}$,
if the calculated angle θ_1 satisfies $\theta_1 > \dfrac{\pi}{2}$, then $\underline{\theta = \pi - \theta_1}$.

Learning Outcomes

Calculate the angle between two straight lines in the coordinate plane using trigonometric slope ratios.

The tangent of the angle between the x—axis and a line is equal to the slope of the line.



Warm Up

Solve the following problems.

1 Find the value of tan105°.

2 Find the angle θ between the two lines $y = 2x+1$ and $y = -3x+2$, where $0 < \theta < \dfrac{\pi}{2}$.

Explanation

1 Since 105° = 45° + 60°:

Convert to forms like 45°±30° × n.

$$\tan(45° + 60°)$$

$$= \frac{\tan 45° + \tan 60°}{1 - \tan 45°\tan 60°}$$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{1+\sqrt{3}}{1 - 1\times\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= \underline{-2-\sqrt{3}}$$

2 Let α and β be the angles that the two lines $y = 2x + 1$ and $y = -3x + 2$ make with the positive x–axis.

$\tan\alpha = 2, \tan\beta = -3$

These are equal to the slopes of the two lines.

Let θ_1 be the angle between the two lines, then $\theta_1 = \alpha - \beta$.

Therefore, $\tan\theta_1 = \tan(\alpha - \beta)$.

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{2-(-3)}{1+2\times(-3)}$$

$$= -1$$

Therefore, $\theta_1 = \frac{3}{4}\pi$

Since $\theta_1 > \frac{\pi}{2}$, $\theta = \pi - \theta_1$

$$= \pi - \frac{3}{4}\pi$$

$$= \underline{\frac{\pi}{4}}.$$

Try

Solve the following problems.

1 Find the value of $\tan\frac{5}{12}\pi$.

2 Find the angle θ between the following two lines, where $0 < \theta < \frac{\pi}{2}$.

(a) $y = \frac{1}{3}x$ and $y = 2x$

(b) $8x + 2y - 3 = 0$ and $-5x + 3y + 1 = 0$

Exercise

Solve the following problems.

1 Find the following values.

(a) $\tan 75°$ **(b)** $\tan 15°$ **(c)** $\tan \frac{13}{12}\pi$

2 Find the angle θ between the following two lines, where $0 < \theta < \frac{\pi}{2}$.

(a) $y = -4x$ and $y = \frac{5}{3}x$ **(b)** $y = (2+\sqrt{3})x$ and $y = x+2$

(c) $y = 3x$ and $y = \frac{1}{2}x+3$ **(d)** $-x+3y = 0$ and $-x + \frac{1}{2}y+2 = 0$

(e) $3x - y+1 = 0$ and $x-2y+4 = 0$ **(f)** $\sqrt{3}x - 2y+3 = 0$ and $3\sqrt{3}x + y+2 = 0$

★ Challenge

A line L_1 passes through the point $P(2,\ 3)$. A second line, L_2, has the equation $x + 2y - 4 = 0$

1. If the acute angle θ between $L_1 and L_2$ satisfies $tan\theta = \frac{3}{4}$,find the two possible equations for line L_1
2. For each possible equation of L_1,calculate the area of the triangle formed by $L_1,\ L_2,\ and the y-$axis.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

5-20

MATHEMATICS

Double Angle Formulas

Chapter 5 Trigonometric Functions

Point!

! Double Angle Formulas

$\sin 2\alpha = \underline{2\sin\alpha\cos\alpha}$ $\qquad \cos 2\alpha = \underline{\cos^2\alpha - \sin^2\alpha}$

$\tan 2\alpha = \underline{\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$ $\qquad = \underline{1-2\sin^2\alpha}$

$= \underline{2\cos^2\alpha - 1}$

! If the values of $\sin 2\alpha$ and $\cos 2\alpha$ are known, use $\tan 2\alpha = \dfrac{\sin 2\alpha}{\cos 2\alpha}$ to find $\tan 2\alpha$.

Learning Outcomes

Apply the double-angle formulas to calculate trigonometric values without using a calculator.

Use if only $\sin\alpha$ is known.

Use if only $\cos\alpha$ is known.

Warm Up

Given $\sin\alpha = \dfrac{1}{3}$, evaluate the following expressions, where α is in Quadrant II.

(a) $\sin 2\alpha$

(b) $\cos 2\alpha$

(c) $\tan 2\alpha$

Explanation

Since $\sin\alpha = \dfrac{1}{3}$, from $\sin^2\alpha + \cos^2\alpha = 1$:

$$\cos^2\alpha = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$$

Since α is in Quadrant II, $\cos\alpha = -\dfrac{2\sqrt{2}}{3}$

Find the value of $\cos\alpha$

(a) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2 \times \frac{1}{3} \times \left(-\frac{2\sqrt{2}}{3}\right) = \underline{-\frac{4\sqrt{2}}{9}}$$

(b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \underline{\frac{7}{9}}$$

(c) $\tan 2\alpha = \dfrac{\sin 2\alpha}{\cos 2\alpha}$

$$= \frac{-\frac{4\sqrt{2}}{9}}{\frac{7}{9}} = \underline{-\frac{4\sqrt{2}}{7}}$$

Try

Given $\cos\alpha = -\frac{3}{5}$, evaluate the following expressions, where $\pi < \alpha < \frac{3}{2}\pi$.

(a) $\sin 2\alpha$ **(b)** $\cos 2\alpha$ **(c)** $\tan 2\alpha$

Exercise

Solve the following problems.

1 Given $\sin\alpha = -\frac{1}{5}$, evaluate the following expressions, where $\pi < \alpha < \frac{3}{2}\pi$.

(a) $\sin 2\alpha$ **(b)** $\cos 2\alpha$ **(c)** $\tan 2\alpha$

2 Given $\cos\alpha = -\frac{2}{3}$, evaluate the following expressions, where $\pi < \alpha < 2\pi$.

(a) $\sin 2\alpha$ **(b)** $\cos 2\alpha$ **(c)** $\tan 2\alpha$

3 Given $\sin\alpha = \frac{5}{13}$, evaluate the following expressions, where α is an obtuse angle.

(a) $\sin 2\alpha$ **(b)** $\cos 2\alpha$ **(c)** $\tan 2\alpha$

4 Given $\cos\alpha = -\frac{1}{4}$, evaluate the following expressions, where α is in Quadrant III.

(a) $\sin 2\alpha$ **(b)** $\cos 2\alpha$ **(c)** $\tan 2\alpha$

★ Challenge

Given $\cos\theta = 0.6$,evaluate $\cos 3\theta \times \tan 2\theta$, where $\pi < \theta < 2\pi$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-21

MATHEMATICS

Half-Angle Formulas

Chapter 5 Trigonometric Functions

Point!

! Do not memorize half-angle formulas; derive them from the double-angle formula for $\cos\alpha$.

$\cos 2\alpha = 1 - 2\sin^2\alpha$ $\quad\xrightarrow{\div 2}\quad$ $\underline{\cos\alpha} = \underline{1 - 2\sin^2\frac{\alpha}{2}}$

$\cos 2\alpha = 2\cos^2\alpha - 1$ $\quad\xrightarrow{\div 2}\quad$ $\underline{\cos\alpha} = \underline{2\cos^2\frac{\alpha}{2} - 1}$

! To find $\sin\frac{\alpha}{2}$ from $\sin^2\frac{\alpha}{2}$, determine the sign based on the quadrant of the angle. (Same for $\cos\frac{\alpha}{2}$)

! To find trigonometric function values for $\frac{n}{8}\pi$, substitute $\frac{n}{4}\pi$ for α in the half-angle formula.

Learning Outcomes

Apply half-angle formulas to calculate trigonometric values without using a calculator.

Warm Up

Solve the following problems.

1 Given $\cos\alpha = \frac{3}{5}$, evaluate the following expressions, where $0 < \alpha < \pi$

(a) $\sin\frac{\alpha}{2}$ **(b)** $\cos\frac{\alpha}{2}$ **(c)** $\tan\frac{\alpha}{2}$

2 Find $\cos\frac{5}{8}\pi$

Explanation

1 **(a)** From $\cos 2\alpha = 1 - 2\sin^2\alpha$:

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$

Therefore, $\frac{3}{5} = 1 - 2\sin^2\frac{\alpha}{2}$

Simplifying gives $\sin^2\frac{\alpha}{2} = \frac{1}{5}$

Since $0 < \alpha < \pi$, $0 < \frac{\alpha}{2} < \frac{\pi}{2}$

Since $\frac{\alpha}{2}$ is in Quadrant I,

$\sin\frac{\alpha}{2} = \underline{\frac{\sqrt{5}}{5}}$

(b) From $\cos 2\alpha = 2\cos^2\alpha - 1$:

$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$

Therefore, $\frac{3}{5} = 2\cos^2\frac{\alpha}{2} - 1$

Simplifying gives $\cos^2\frac{\alpha}{2} = \frac{4}{5}$

Since $0 < \frac{\alpha}{2} < \frac{\pi}{2}$, $\cos\frac{\alpha}{2} = \underline{\frac{2\sqrt{5}}{5}}$

Determine the sign based on the quadrant.

(c) Since the values of $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ are known, use $\tan\frac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$.

$$\tan\frac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \dfrac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}}$$

$$= \frac{\sqrt{5}}{2\sqrt{5}} \qquad = \frac{1}{2}$$

2 Since $\cos\frac{5}{8}\pi$ is a trigonometric function of $\frac{n}{8}\pi$, use the half-angle formula.

From $\cos 2\alpha = 2\cos^2\alpha - 1$,

we get $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$

Substitute $\alpha = \frac{5}{4}\pi$ into this equation:

$$\cos\frac{5}{4}\pi = 2\cos^2\frac{5}{8}\pi - 1$$

$$-\frac{1}{\sqrt{2}} = 2\cos^2\frac{5}{8}\pi - 1$$

Simplifying gives $\cos^2\frac{5}{8}\pi = \frac{2-\sqrt{2}}{4}$

Since $\frac{\pi}{2} < \frac{5}{8}\pi < \frac{3}{2}\pi$, $\cos\frac{5}{8}\pi = \underline{-\frac{\sqrt{2-\sqrt{2}}}{2}}$

Try

Solve the following problems.

1 Given $\cos\alpha = -\frac{1}{3}$, evaluate the following expressions, where $0 < \alpha < \pi$.

(a) $\sin\frac{\alpha}{2}$ **(b)** $\cos\frac{\alpha}{2}$ **(c)** $\tan\frac{\alpha}{2}$

2 Evaluate $\sin\frac{3}{8}\pi$.

Exercise

Solve the following problems.

1 Given $\cos\alpha = \frac{5}{7}$, evaluate the following expressions, where $0 < \alpha < \pi$.

(a) $\sin\frac{\alpha}{2}$ **(b)** $\cos\frac{\alpha}{2}$ **(c)** $\tan\frac{\alpha}{2}$

2 Given $\cos\alpha = -\frac{3}{5}$, evaluate the following expressions, where $2\pi < \alpha < 3\pi$.

(a) $\sin\frac{\alpha}{2}$ **(b)** $\cos\frac{\alpha}{2}$ **(c)** $\tan\frac{\alpha}{2}$

3 Given $\cos\alpha = -\frac{2}{3}$, evaluate the following expressions, where $3\pi < \alpha < 4\pi$.

(a) $\sin\frac{\alpha}{2}$ **(b)** $\cos\frac{\alpha}{2}$ **(c)** $\tan\frac{\alpha}{2}$

4 Evaluate the following values.

(a) $\cos\frac{3}{8}\pi$ **(b)** $\sin\frac{5}{8}\pi$ **(c)** $\cos\frac{\pi}{8}$

★ Challenge

Given $tanA = \frac{1}{2}\sqrt{5}$,where

$4\pi < \mathrm{A} < 5\pi,\ and cosB = -\frac{1}{3}\sqrt{5}$

,where $5\pi < \mathrm{B} < 6\pi$,

evaluate:

$a)cos\left(\frac{A}{2} - \frac{B}{2}\right)\ b)sin\left(\frac{A}{4}\right)$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

5-22

MATHEMATICS

Equations and Inequalities Involving Double Angles

Chapter 5 Trigonometric Functions

Learning Outcomes

Solve equations and inequalities involving double angles.

Point!

! In equations and inequalities containing 2θ, use the double-angle formulas to express the trigonometric functions in terms of the same angle.

• Double Angle Formulas.

$\sin 2\theta =$ $\underline{2\sin\theta\cos\theta}$ $\qquad \cos 2\theta =$ $\underline{1-2\sin^2\theta}$

$= \underline{2\cos^2\theta - 1}$

Use to rewrite as an expression in terms of only $\sin\theta$.

Use to rewrite as an expression in terms of only $\cos\theta$.

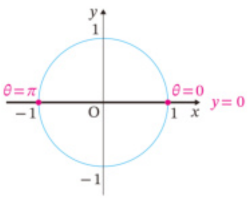
Warm Up

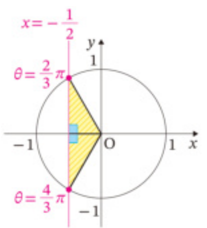
Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\sin 2\theta + \sin\theta = 0$ **(b)** $\cos 2\theta - 7\cos\theta + 4 \geq 0$

Explanation

(a)





$\sin 2\theta + \sin\theta = 0$

$2\sin\theta\cos\theta + \sin\theta = 0$

$\sin\theta(2\cos\theta + 1) = 0$

This implies that $\sin\theta = 0$ or $2\cos\theta + 1 = 0$

If $\sin\theta = 0$, then $\theta = 0$ or π

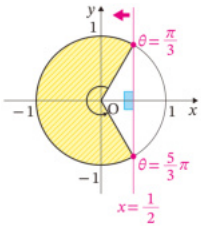
If $2\cos\theta + 1 = 0$, i.e., $\cos\theta = -\frac{1}{2}$, then $\theta = \frac{2}{3}\pi$ or $\frac{4}{3}\pi$

Therefore, $\theta = 0$ or $\frac{2}{3}\pi$ or π or $\frac{4}{3}\pi$.

Use double angle formula to express the trigonometric functions in terms of the same angle.

If $AB = 0$, $A = 0$ or $B = 0$

(b)



$\cos 2\theta - 7\cos\theta + 4 \geq 0$

$(2\cos^2\theta - 1) - 7\cos\theta + 4 \geq 0$

$2\cos^2\theta - 7\cos\theta + 3 \geq 0$

Solving this gives $\cos\theta \leq \frac{1}{2}$ or $\cos\theta \geq 3$ ...... (i)

Also, $-1 \leq \cos\theta \leq 1$ ......(ii)

From(i) and (ii), $-1 \leq \cos\theta \leq \frac{1}{2}$

From the figure on the left, the required solution is

$\underline{\frac{\pi}{3} \leq \theta \leq \frac{5}{3}\pi.}$

Use double angle formula to express the trigonometric functions in terms of the same angle.

Try

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

1. $\sin 2\theta + \cos\theta = 0$
2. $\cos 2\theta + \sin\theta = 0$
3. $\cos 2\theta - 3\cos\theta - 1 \leq 0$

Exercise

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

1 $\sin 2\theta = \sin\theta$

2 $\sin 2\theta - \cos\theta = 0$

2 $\sin 2\theta = \sqrt{3}\sin\theta$

4 $\cos 2\theta = 3\cos\theta + 1$

5 $\cos 2\theta + \cos\theta + 1 = 0$

6 $\cos 2\theta = -5\cos\theta - 3$

7 $\cos 2\theta - \sin\theta \geq 0$

8 $2\cos 2\theta + 4\cos\theta - 1 \geq 0$

9 $\cos 2\theta < -\sin\theta$

★ Challenge

Solve the following inequality for θ,where $0 \leq \theta < 2\pi$:

$\cos 2\theta \geq \cos\theta$

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

5-23

MATHEMATICS

Harmonic Addition (Trigonometric Synthesis)

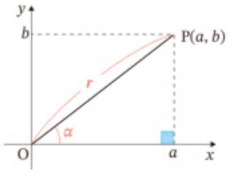
Chapter 5 Trigonometric Functions

Point!

❗ $a\sin\theta + b\cos\theta$ can be rewritten in the following form:

$a\sin\theta + b\cos\theta = r\sin(\theta + \alpha)$

❗ Steps for Transformation.



❶ Plot point P **(a,b)** on the coordinate plane.

❷ Find **the length r of OP**.

❸ Draw a reference right triangle with hypotenuse OP and find the **angle α whose terminal side is OP**.

❹ Substitute obtained values of r and α into **$r\sin(\theta + \alpha)$**.

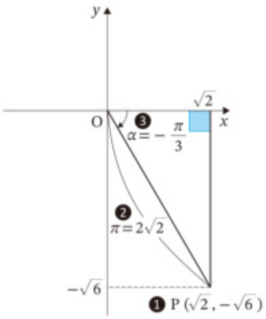
Learning Outcomes

Employ the harmonic addition method (for two terms with the same frequency θ but different amplitudes a and b) to transform trigonometric expressions composed of sin and cos into a single equivalent trigonometric function.

Warm Up

Rewrite $\sqrt{2}\sin\theta - \sqrt{6}\cos\theta$ in the form $r\sin(\theta+\alpha)$, where $r>0$ and $-\pi<\alpha<\pi$.

Explanation



Plot point P $(\sqrt{2}, -\sqrt{6})$ on the coordinate plane.

Let r be the length of OP.

$$r = \sqrt{(\sqrt{2})^2 + (-\sqrt{6})^2}$$

$$= 2\sqrt{2}$$

From the figure on the left,

the value of α in the range $-\pi < \alpha < \pi$ is:

$$\alpha = -\frac{\pi}{3}$$

Therefore,

$$\sqrt{2}\sin\theta - \sqrt{6}\cos\theta = 2\sqrt{2}\sin\left(\theta - \frac{\pi}{3}\right)$$

❶ Plot point P(a, b) on the coordinate plane.

❷ Find the length r of OP.

❸ Draw a reference right triangle with hypotenuse OP and find α.

❹ Substitute obtained values of r and α into $r\sin(\theta + \alpha)$.

Try

Rewrite the following expressions in the form $r\sin(\theta + \alpha)$, where $r > 0$ and $-\pi < \alpha < \pi$.

(a) $\sin\theta + \sqrt{3}\cos\theta$

(b) $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta$

Exercise

Rewrite the following expressions in the form $r\sin(\theta + \alpha)$, where $r > 0$ and $-\pi < \alpha < \pi$.

(a) $\sqrt{3}\sin\theta - \cos\theta$

(b) $\sin\theta + \cos\theta$

(c) $\sqrt{3}\sin\theta + \cos\theta$

(d) $\sqrt{3}\sin\theta - \sqrt{3}\cos\theta$

(e) $\sqrt{6}\sin\theta - \sqrt{2}\cos\theta$

(f) $2\sin\theta + 2\cos\theta$

★ Challenge

Rewrite the following expression in the form:

$r\sin(\theta + \alpha)$, where $r > 0$ *and* $-\pi < \alpha < \pi$

$\sqrt{2}\sin\theta - \sqrt{6}\cos\theta$

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

5-24
MATHEMATICS

Equations and Inequalities Using Harmonic Addition

Chapter 5 Trigonometric Functions

Point!

- If an equation or inequality contains the form $a\sin\theta + b\cos\theta$, rewrite it in the form $r\sin(\theta + \alpha)$.
- When solving equations/inequalities with synthesized trigonometric functions, check the range of the angle after synthesis and solve within that range.

Learning Outcomes

Employ the harmonic addition method (for two terms with the same frequency θ but different amplitudes a and b) to transform trigonometric expressions composed of sine and cosine into a single equivalent trigonometric function, and use this to solve equations.

Warm Up

Solve the following equations and inequalities for θ, where $0 \leq \theta < 2\pi$.

(a) $\sin\theta - \cos\theta = 1$ **(b)** $\sqrt{3}\sin\theta + \cos\theta < \sqrt{2}$

See 5-16.

Explanation

(a) $\sin\theta - \cos\theta = 1$

$$\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = 1$$

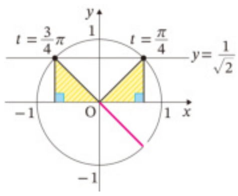
$$\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Rewrite the LHS in the form $r\sin(\theta + \alpha)$.

Let $\theta - \frac{\pi}{4} = t$, then $-\frac{\pi}{4} \leq t < \frac{7}{4}\pi$.

Check the range of the new variable.

The solutions for $\sin t = \frac{1}{\sqrt{2}}$ in this range are: $t = \frac{\pi}{4}$ or $\frac{3}{4}\pi$.



Since $-\frac{\pi}{4} \leq t < \frac{7}{4}\pi$, consider one full revolution from here.

Therefore, $\theta - \frac{\pi}{4} = \frac{\pi}{4}$ or $\frac{3}{4}\pi$

$$\underline{\theta = \frac{\pi}{2} \text{ or } \pi}$$

(b) $\sqrt{3}\sin\theta + \cos\theta < \sqrt{2}$

$2\sin\left(\theta + \frac{\pi}{6}\right) < \sqrt{2}$

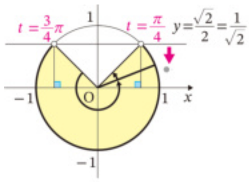
$\sin\left(\theta + \frac{\pi}{6}\right) < \frac{\sqrt{2}}{2}$

Let $\theta + \frac{\pi}{6} = t$, then $\frac{\pi}{6} \le t < \frac{13}{6}\pi$.

Rewrite the LHS in the form $r\sin(\theta + \alpha)$.

Since $\frac{\pi}{6} \le t < \frac{13}{6}$, consider one full revolution from here.

The solutions for $\sin t < \frac{\sqrt{2}}{2}$ in this range are:



$\frac{\pi}{6} \le t < \frac{\pi}{4}$ or $\frac{3}{4}\pi < t < \frac{13}{6}\pi$

Therefore, $\frac{\pi}{6} \le \theta + \frac{\pi}{6} < \frac{\pi}{4}$ or $\frac{3}{4}\pi < \theta + \frac{\pi}{6} < \frac{13}{6}\pi$

Solving for θ yields:

$0 \le \theta < \frac{\pi}{12}$ or $\frac{7}{12}\pi < \theta < 2\pi$

Try

Solve the following equations and inequalities for θ, where $0 \le \theta < 2\pi$.

(a) $\sin\theta + \cos\theta + 1 = 0$

(b) $\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$

(c) $\sqrt{3}\sin\theta \le \cos\theta$

Exercise

Work with a partner

Collaborate with your group members to answer the Try section, then discuss the answer.

Solve the following equations and inequalities for θ, where $0 \le \theta < 2\pi$.

(a) $\sin\theta - \cos\theta = -1$

(b) $\sin\theta + \cos\theta = \frac{\sqrt{2}}{2}$

(c) $2(\sin\theta + \cos\theta) = \sqrt{6}$

(d) $\sqrt{3}\sin\theta + \cos\theta = \sqrt{2}$

(e) $\sin\theta - \sqrt{3}\cos\theta = \sqrt{6}$

(f) $\sqrt{3}\sin\theta - \cos\theta = 1$

(g) $\sin\theta \ge \sqrt{3}\cos\theta$

(h) $\sqrt{3}\sin\theta + \cos\theta < -1$

(i) $\sin\theta - \sqrt{3}\cos\theta > 1$

★ Challenge

Solve the following inequality for θ, where $0 \le \theta < 2\pi$:

$$\frac{(\sqrt{6}\sin\theta - \sqrt{2}\cos\theta)^2 - 2(\sqrt{6}\sin\theta - \sqrt{2}\cos\theta) - 2}{\sqrt{6}\sin\theta - \sqrt{2}\cos\theta + 3\sqrt{2}} \le 0$$

5-25

MATHEMATICS

Maximum and Minimum of Trigonometric Functions (2)

Chapter 5 Trigonometric Functions

Point!

! To find the maximum and minimum of trigonometric functions, use the same transformations as when solving equations and inequalities.

• If an expression contains the form $a\sin\theta + b\cos\theta$, rewrite it in the form $r\sin(\theta + \alpha)$.

• If trigonometric functions involve double angles, use the double angle formulas to express the trigonometric functions in terms of the same angle.

! To find maximum and minimum, pay attention to the ranges $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$

Learning Outcomes

Employ the harmonic addition method (for two terms with the same frequency θ but different amplitudes a and b) to transform trigonometric expressions composed of sine and cosine into a single equivalent trigonometric function, and use this to solve equations and determine absolute maximum values in wave systems.

Warm Up

Solve the following problems.

1 Find the maximum and minimum values of $y = \sin\theta - \sqrt{3}\cos\theta$.

See 5-24.

2 Find the maximum and minimum values of the function $y = 2\cos 2\theta + 4\cos\theta + 1,\ 0 \leq \theta < 2\pi$. Also, find the corresponding values of θ.

See 5-22.

Explanation

1 $y = \sin\theta - \sqrt{3}\cos\theta$

$= 2\sin\left(\theta - \frac{\pi}{3}\right)$

Since $-1 \leq \sin(\theta - \frac{\pi}{3}) \leq 1$:

$-2 \leq 2\sin(\theta - \frac{\pi}{3}) \leq 2$

Therefore, maximum value of 2; minimum value of −2

Rewrite the RHS in the form $r\sin(\theta + \alpha)$.

To find the range of $y = 2\sin(\theta - \frac{\pi}{3})$, construct the inequality the form of $\leq 2\sin(\theta - \frac{\pi}{3}) \leq$.

2 $y = 2\cos 2\theta + 4\cos\theta + 1$

$= 2(2\cos^2\theta - 1) + 4\cos\theta + 1$

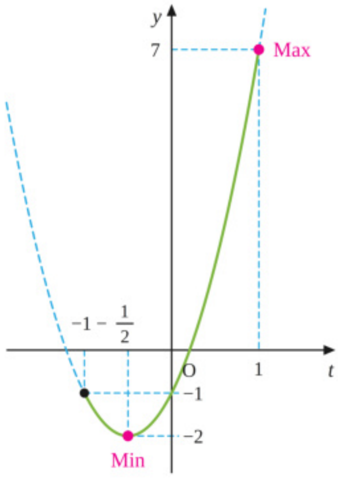
$= 4\cos^2\theta + 4\cos\theta - 1$

Let $\cos\theta = t$, then $-1 \le t \le 1$

$y = 4t^2 + 4t - 1$

$= 4\left(t + \frac{1}{2}\right)^2 - 2$

Drawing this graph gives the figure on the right.



From the graph:

Maximum value of 7 for $t = 1$

If $t = 1$, $\cos\theta = 1$, so $\theta = 0$

Also, minimum value of -2 for $t = -\frac{1}{2}$.

If $t = -\frac{1}{2}$, $\cos\theta = -\frac{1}{2}$, so $\theta = \frac{2}{3}\pi$ or $\frac{4}{3}\pi$.

Therefore, maximum value of 7 for $\theta = 0$;

minimum value of -2 for $\theta = \frac{2}{3}\pi$ or $\frac{4}{3}\pi$

Use double angle formula to express the trigonometric functions in terms of the same angle.

Check the range after substitution (see 5 – 17).

Try

Solve the following problems.

1 Find the maximum and minimum values of $y = -\sqrt{3}\sin\theta + \cos\theta$.

2 Find the maximum and minimum values of the function $y = \cos 2\theta - 2\cos\theta$, where $0 \le \theta < 2\pi$. Also, find the corresponding values of θ

Exercise

Solve the following problems.

1 Find the maximum and minimum values of the following functions.

(a) $y = \sin\theta - \sqrt{3}\cos\theta - 1$ **(b)** $y = \sin\theta + \cos\theta$

(c) $y = 2\sin\theta + 2\cos\theta$

2 Find the maximum and minimum values of the following functions, where $0 \le \theta < 2\pi$. Also, find the corresponding values of θ.

(a) $y = 2\sin\theta - \cos 2\theta$ **(b)** $y = 2\cos 2\theta + 4\sin\theta - 1$

(c) $y = \cos 2\theta + 2\sqrt{3}\cos\theta$

★ Challenge

Find the maximum and minimum values of

$y = |\sin\theta - 2\cos\theta|$, where $0 \le \theta < 2\pi$

Also, find the corresponding values of θ

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Reflect

When you see a complex trigonometric expression containing both single angles (θ)and double angles (2θ),how do you decide whether to expand the double angles down or compress the single angles up? What structural cues guide your choice?

6-1

MATHEMATICS

Laws of Exponents for Integer Exponents

Chapter 6 Exponents and Logarithms

Concept 1 · Laws of Exponents and Roots

In epidemiology and pharmacology, understanding how microscopic entities replicate or how medication clears from the human body is essential. For instance, when a bacterium like Escherichia coli enters a nutrient-rich environment, it divides by binary fission every 20 minutes.

This rapid duplication means a population grows by a factor of 2^n over n cycles. Conversely, when an antibiotic is administered, its concentration in the bloodstream decays at a fractional rate over time. Mathematically capturing these rapid scales—whether tracking billions of microbes or microscopic milligrams of life-saving medicine—requires a precise framework. Guiding Question: How do the algebraic rules of fractional exponents make it possible to move smoothly from modeling growth in discrete steps, such as 2^n, to representing growth across continuous intervals, such as $\sqrt[n]{2}$?

Learning Outcomes

Applies the laws of integer exponents in simplifying compound algebraic expressions.

Point!

! When written as a^n,a is called the **base** , and n is called the **exponent**

If a = 0 and n is a positive integer:

$a^0 = 1$

$a^{-n} = \left(\frac{1}{\boldsymbol{a}}\right)^n$

Example: $a^{-2} = \left(\frac{1}{a}\right)^2 \left(\frac{1}{a}\right)^{-3} = \mathbf{a}^3$

A power with an index of 0 is 1

A negative n–th power is the n–th power of the reciprocal.

Laws of Exponents

Let $a \neq 0, b \neq 0$, and let m, n be integers.

$a^m \times a^n = a^{m+n}$ Example: $a^3 \times a^4 = \mathbf{a}^7$

$a^m \div a^n = a^{m-n}$ Example: $a^5 \div a^3 = \mathbf{a}^2$

$(a^m)^n = a^{mn}$ Example: $(a^3)^2 = \mathbf{a}^6$

$(ab)^n = a^n b^n$ Example: $(a^2 b)^3 = \mathbf{a}^6 b^3$

Warm Up

Solve the following problems.

1 Find the following values.

(a) 10^0 **(b)** 2^{-3} **(c)** $(0.2)^{-2}$

2 Let $a \neq 0$ and $b \neq 0$. Perform the following calculations.

(a) $a^{-5} \times a^8$ **(b)** $a^5 \div a^{-3}$ **(c)** $(a^{-1}b)^2$ **(d)** $(2^2)^3 \div 2^{-2} \times 2^{-3}$

Explanation

1 **(a)** $10^0 = 1$

A power with an exponent of 0 is 1

(b) $2^{-3} = \left(\frac{1}{2}\right)^3$

$= \frac{1}{8}$

(c) $(0.2)^{-2} = \left(\frac{1}{5}\right)^{-2}$

$= 5^2$

$= 25$

2 **(a)** $a^{-5} \times a^8 = a^{-5+8}$

$= a^3$

(b) $a^5 \div a^{-3} = a^{5-(-3)}$

$= a^8$

(c) $(a^{-1}b)^2 = a^{-2}b^2$

$= \left(\frac{1}{a}\right)^2 b^2$

$= \frac{b^2}{a^2}$

(d) $(2^2)^3 \div 2^{-2} \times 2^{-3} = 2^6 \div 2^{-2} \times 2^{-3}$

$= 2^{6-(-2)+(-3)}$

$= 2^5$

$= 32$

Try

Solve the following problems.

1 Find the following values.

(a) 5^0 **(b)** 3^{-2} **(c)** $(0.25)^{-2}$

A power of −3 is the cube of the reciprocal.

Convert decimals to fractions before calculating.

2 Let $a \neq 0$ and $b \neq 0$ Perform the following calculations.

Add the exponents for multiplication.

Subtract the exponents when dividing.

(a) $a^5 \times a^4$ **(b)** $a^3 \div a^{-2}$

(c) $(a^{-2}b^4)^{-1}$ **(d)** $2^3 \times 2^{-7}$

(e) $5^{-4} \div 5^{-6}$ **(f)** $(a^3b^{-1})^3 \times (a^2b^{-3})^{-2}$

(g) $4^{-5} \times 4^2 \div 4^{-2}$ **(h)** $2^4 \div (2^{-1})^3 \times 2^{-2}$

Exercise

Solve the following problems.

1 Find the following values.

(a) 3^0

(b) 11^0

(c) $\left(\frac{2}{3}\right)^0$

(d) 4^{-3}

(e) 5^{-3}

(f) 6^{-2}

(g) $\left(\frac{1}{3}\right)^{-3}$

(h) $\left(\frac{3}{5}\right)^{-2}$

(i) $(0.04)^{-2}$

2 Let $a \neq 0$ and $b \neq 0$ Perform the following calculations.

(a) $a^2 \times a^4$

(b) $a^{-2} \times a^5$

(c) $a^{-3} \times a^{-4}$

(d) $a^3 \div a^5$

(e) $a^8 \div a^{-3}$

(f) $a^{-6} \div a^{-2}$

(g) $(a^{-1})^3$

(h) $(a^3b^4)^{-2}$

(i) $(a^{-3}b^2)^{-3}$

(j) $3^5 \times 3^{-3}$

(k) $4^4 \times 4^{-3}$

(l) $5^5 \times 5^{-5}$

3 Let $a \neq 0$ and $b \neq 0$ Perform the following calculations.

(a) $5^4 \div 5^2$

(b) $3^2 \div 3^{-2}$

(c) $2^{-4} \div 2^{-5}$

(d) $(a^{-2}b)^3 \times (a^2b^{-1})^2$

(e) $(a^{-3}b)^{-3} \times (a^4b^2)^{-2}$

(f) $(2a^3b^{-1})^3 \times \left(-3ab^{-2}\right)^{-3}$

(g) $3^2 \times 3^{-1} \div 3^3$

(h) $5^{-4} \times 5^5 \div 5^{-2}$

(i) $2^4 \times 2^{-8} \div 2^{-6}$

(j) $5^3 \times \left(5^{-2}\right)^2 \div 5^{-4}$

(k) $3^5 \div \left(3^{-2}\right)^{-3} \times 3$

(l) $\left(\frac{a}{2}\right)^3 \times a^{-4} \div (2a)^{-2}$

Work with a partner

Collaborate with your group members to answer question 2 parts [e:h], then discuss the answer.

★ Challenge

Let $a \neq 0$ and $b \neq 0$. Simplify the following expression completely and write your answer with positive exponents only:

$$\frac{\left(\frac{2a^2}{b^{-3}}\right)^{-2} \times \left(4a^{-3}b^2\right)^3 \div \left(\frac{a}{2b}\right)^{-4}}{a^{-2}b^2}$$

6-2

MATHEMATICS

Roots

Chapter 6 Exponents and Logarithms

Point!

!

If n is an integer, a number that becomes a when raised to the $n-$th power is called the $n-$th root of a.

The 2nd root, 3rd root, ... of a are collectively called the roots of a.

Example: Since $2^3 = 8$, 2 is the 3rd root of 8

If a>0, the positive $n-$th root of a is represented as $\sqrt[n]{a}$. (However, if $n = 2$, it is abbreviated as $\sqrt{a}$.)

$(\sqrt[n]{a})^n = \underline{a}$, $\sqrt[n]{a^n} = \underline{a}$

$\sqrt[n]{a^n \times b} = \underline{a}\sqrt[n]{b}$

Properties of roots: For a>0, b >0, let m and n be positive integers.

$\sqrt[n]{a}\times\sqrt[n]{b} = {}^{n}\sqrt{ab}$, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Learning Outcomes

Connects the concept of the root with the concept of the exponent for mathematical expressions.

By taking the $n-$th power, you can remove the $\sqrt[n]{\ }$

Warm Up

Solve the following problems.

1 Find the following values.

(a) $\sqrt[3]{64}$ **(b)** $\sqrt[3]{\dfrac{1}{27}}$

2 Perform the following calculations.

(a) $\sqrt[3]{12}\times\sqrt[3]{18}$ **(b)** $\dfrac{\sqrt[3]{250}}{\sqrt[3]{2}}$ **(c)** $(\sqrt[6]{8})^2$

(d) $\sqrt[3]{\sqrt{729}}$ **(e)** $\sqrt[3]{54}-\sqrt[3]{16}+\sqrt[3]{2}$

Explanation

1 (a) $\sqrt[3]{64}$

$= \sqrt[3]{2^6}$

$= \sqrt[3]{(2^2)^3}$

$= 2^2$

$= 4$

(b) $\sqrt[3]{\frac{1}{27}}$

$= \sqrt[3]{\left(\frac{1}{3}\right)^3}$

$= \frac{1}{3}$

Prime factorize the number inside the $\sqrt{\ }$

Look for perfect cubes.

$\sqrt[n]{a^n} = a$

2 (a) $\sqrt[3]{(2^2 \times 3)} \times \sqrt[3]{(2 \times 3^2)}$

$= \sqrt[3]{(2^3 \times 3^3)}$

$= 6$

(b) $\frac{\sqrt[3]{250}}{\sqrt[3]{2}}$

$= \sqrt[3]{\frac{250}{2}}$

$= \sqrt[3]{125}$

$= \sqrt[3]{5^3}$

$= 5$

(c) $(\sqrt[6]{8})^2$

$= (\sqrt[6]{2^3})^2$

$= \sqrt[6]{2^{3\times 2}}$

$= \sqrt[6]{2^6}$

$= 2$

(d) $\sqrt[3]{\sqrt{729}}$

$= \sqrt[3\times 2]{729}$

$= \sqrt[6]{729}$

$= \sqrt[6]{3^6}$

$= 3$

(e) $\sqrt[3]{54} - \sqrt[3]{16} + \sqrt[3]{2}$

$= \sqrt[3]{3^3} \times 2 - \sqrt[3]{2^3} \times 2 + \sqrt[3]{2}$

$= 3\sqrt[3]{2} - 2\sqrt[3]{2} + \sqrt[3]{2}$

$= 2\sqrt[3]{2}$

Look for perfect cubes.

$\sqrt[n]{a^n} = a$

Use prime factorisation.

$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Use prime factorisation.

Use prime factorisation.

Try

Solve the following problems.

1 Find the following values.

(a) $\sqrt[4]{81}$ (b) $\sqrt[4]{\frac{1}{16}}$

2 Perform the following calculations.

(a) $\sqrt[4]{8} \times \sqrt[4]{32}$ (b) $\frac{\sqrt[4]{48}}{\sqrt[4]{3}}$ (c) $(\sqrt[6]{100})^3$

(d) $\sqrt{\sqrt[3]{64}}$ (e) $\sqrt[3]{81} - \sqrt[3]{3} + \sqrt[3]{24}$

Exercise

Solve the following problems.

1 Find the following values.

(a) $\sqrt[3]{216}$ (b) $\sqrt[4]{625}$ (c) $\sqrt[5]{243}$

(d) $\sqrt[3]{\frac{1}{64}}$ (e) $\sqrt[3]{\frac{1}{125}}$ (f) $\sqrt[4]{\frac{1}{81}}$

(g) $\sqrt[3]{1}$ (h) $\sqrt[5]{100000}$ (i) $\sqrt[4]{0.0001}$

2 Perform the following calculations.

(a) $\sqrt[3]{2} \times \sqrt[3]{4}$ (b) $\sqrt[3]{32} \times \sqrt[3]{2}$ (c) $\sqrt[4]{27} \times \sqrt[4]{243}$

(d) $\dfrac{\sqrt[3]{81}}{\sqrt[3]{3}}$ (e) $\dfrac{\sqrt[3]{2}}{\sqrt[3]{128}}$ (f) $\dfrac{\sqrt[3]{5}}{\sqrt[3]{625}}$

(g) $(\sqrt[4]{49})^2$ (h) $(\sqrt[6]{27})^2$ (i) $(\sqrt[8]{16})^4$

(j) $\sqrt{\sqrt[3]{3^{12}}}$ (k) $\sqrt[5]{1024}$ (l) $\sqrt{\sqrt[3]{729}}$

(m) $\sqrt[3]{24} + \sqrt[3]{81} - 4\sqrt[3]{3}$ (n) $\sqrt[4]{32} + \sqrt[4]{2} - \sqrt[4]{162}$ (o) $\sqrt[3]{16} + \sqrt[3]{2} - \sqrt[3]{54}$

★ Challenge

Perform the following calculation:

$$\sqrt[3]{\sqrt[4]{4096}} \div \sqrt{\sqrt[6]{4096}}$$

$(\sqrt[n]{a})^m = \sqrt[n]{a^m}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Use prime factorisation.

Use prime factorisation.

$\sqrt[n]{a^n \times b} = a\sqrt[n]{b}$

Terms with the same $\sqrt[n]{\ }$ can be combined.

Work with a partner

Collaborate with your group members to answer question 2 parts d, e, then discuss the answer.

6-3

MATHEMATICS

Rational Laws of Indices (1)

Chapter 6 Exponents and Logarithms

Learning Outcomes

Applies the laws of rational (fractional) exponents in simplifying compound algebraic expressions.

Point!

!

If $a > 0$ and $m,\ n$ are positive integers:

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad (\sqrt[n]{a})^{m} = a^{\frac{m}{n}} \quad \sqrt[n]{a^{m}} = a^{\frac{m}{n}}$$

Example: $\sqrt[3]{a} = a^{\frac{1}{3}}$ Example: $(\sqrt[4]{a})^{3} = a^{\frac{3}{4}}$ Example: $\sqrt[3]{a^{2}} = a^{\frac{2}{3}}$

The laws of exponents hold true even if the exponents are fractions. Let $a > 0$, and $m,\ n$ be fractions.

$$(a^{m})^{n} = a^{mn} \quad \text{Example: } \left(a^{\frac{1}{2}}\right)^{\frac{3}{5}} = a^{\frac{1}{2}\times\frac{3}{5}} = a^{\frac{3}{10}}$$

$$\left(\frac{1}{a}\right)^{n} = \frac{1}{a^{n}} = a^{-n} \quad \text{Example: } \frac{1}{a^{\frac{1}{5}}} = a^{-\frac{1}{5}}$$

Warm Up

Solve the following problems.

1 Express the following expressions in the form a^{r}

(a) $\sqrt{5}$ **(b)** $\sqrt[5]{2^{4}}$ **(c)** $\frac{1}{\sqrt[5]{3^{3}}}$

2 Find the following values.

(a) $27^{\frac{2}{3}}$ **(b)** $64^{-\frac{3}{2}}$ **(c)** $\left(\frac{25}{4}\right)^{-\frac{3}{2}}$ **(d)** $\left[\left(\frac{16}{9}\right)^{-\frac{3}{4}}\right]^{\frac{2}{3}}$

Explanation

1 (a) $\sqrt{5}$

(b) $\sqrt[5]{2^4}$

$= 2^{\frac{4}{5}}$

(c) $\dfrac{1}{\sqrt[5]{3^3}}$

$= \dfrac{1}{3^{\frac{3}{5}}}$

$= 3^{-\frac{3}{5}}$

$\sqrt{\ } = \sqrt[2]{\ }$

Express the root in the form a^r

$\frac{1}{a^n} = a^{-n}$

2 (a) $27^{\frac{2}{3}}$

$= (3^3)^{\frac{2}{3}}$

$= 3^{3\times\frac{2}{3}}$

$= 3^2$

$= 9$

(b) $64^{-\frac{3}{2}}$

$= (2^6)^{-\frac{3}{2}}$

$= 2^{6\times\left(-\frac{3}{2}\right)}$

$= 2^{-9}$

Prime factorise the base to make it smaller.

(c) $\left(\dfrac{25}{4}\right)^{-\frac{3}{2}}$

$= \left[\left(\dfrac{5}{2}\right)^2\right]^{-\frac{3}{2}}$

$= \left(\dfrac{5}{2}\right)^{-3}$

$= \dfrac{8}{125}$

(d) $\left[\left(\dfrac{16}{9}\right)^{-\frac{3}{4}}\right]^{\frac{2}{3}}$

$= \left[\left[\left(\dfrac{4}{3}\right)^2\right]^{-\frac{3}{4}}\right]^{\frac{2}{3}}$

$= \left(\dfrac{4}{3}\right)^{2\times\left(-\frac{3}{4}\right)\times\frac{2}{3}}$

$= \left(\dfrac{4}{3}\right)^{-1}$

$= \dfrac{3}{4}$

Try

Solve the following problems.

1 Express the following expressions in the form a^r

(a) $\sqrt{a}$ (b) $\sqrt[3]{a^5}$ (c) $\dfrac{1}{(\sqrt{a})^3}$

2 Find the following values.

(a) $81^{\frac{3}{4}}$ (b) $16^{-\frac{3}{4}}$ (c) $\left(\dfrac{25}{64}\right)^{-\frac{3}{2}}$ (d) $\left[\left(\dfrac{27}{125}\right)^{\frac{1}{2}}\right]^{-\frac{4}{3}}$

Exercise

Solve the following problems.

1 Express the following expressions in the form a^r

(a) $\sqrt{2}$ **(b)** $\sqrt{6}$ **(c)** $\sqrt{7}$

(d) $\sqrt[4]{a^3}$ **(e)** $\sqrt[6]{a^5}$ **(f)** $\sqrt[3]{2^2}$

(g) $\dfrac{1}{\sqrt{a}}$ **(h)** $\dfrac{1}{\sqrt[3]{a^2}}$ **(i)** $\dfrac{1}{\sqrt[4]{5^3}}$

2 Find the following values.

(a) $25^{\frac{3}{2}}$ **(b)** $16^{\frac{3}{4}}$ **(c)** $64^{\frac{2}{3}}$

(d) $9^{-\frac{3}{2}}$ **(e)** $125^{-\frac{2}{3}}$ **(f)** $81^{-\frac{3}{4}}$

(g) $\left(\dfrac{1}{8}\right)^{-\frac{1}{3}}$ **(h)** $\left(\dfrac{125}{8}\right)^{-\frac{2}{3}}$ **(i)** $\left(\dfrac{16}{81}\right)^{-\frac{3}{4}}$

(j) $\left[\left(\dfrac{1}{25}\right)^{\frac{2}{5}}\right]^{-\frac{5}{4}}$ **(k)** $\left[\left(\dfrac{81}{100}\right)^{-\frac{3}{4}}\right]^{\frac{2}{3}}$ **(l)** $\left[\left(\dfrac{64}{27}\right)^{-\frac{4}{3}}\right]^{-\frac{1}{2}}$

★ Challenge

Simplify the following expression completely and write your answer as a fully reduced fraction:

$$\left(\left(\left(\frac{1000}{27}\right)^{\frac{-2}{3}}\right)^{\frac{7}{15}}\right)^{\frac{-15}{14}}$$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

6-4

MATHEMATICS

Rational Laws of Indices (2)

Chapter 6 Exponents and Logarithms

Point!

!

The laws of exponents hold true even if the exponents are fractions.

Let $a > 0$, $b > 0$, and $m,\ n$ be fractions.

$a^m \times a^n = a^{m+n}$ $\quad a^m \div a^n = a^{m-n}$ $\quad (a^m)^n = a^{mn}$ $\quad (ab)^n = a^n b^n$

Learning Outcomes

Applies the laws of rational (fractional) exponents in simplifying compound algebraic expressions.

Warm Up

Perform the following calculations.

(a) $2^{\frac{2}{3}} \times 2^{\frac{1}{2}} \div 2^{\frac{7}{6}}$ **(b)** $6^{\frac{1}{2}} \times 36^{\frac{1}{4}}$

(c) $\sqrt[3]{6} \times \sqrt{6} \times \sqrt[6]{6}$ **(d)** $\sqrt{2} \div \sqrt[6]{32} \times \sqrt[3]{16}$

Explanation

Make the bases the same.

(a) $2^{\frac{2}{3}} \times 2^{\frac{1}{2}} \div 2^{\frac{7}{6}}$

$= 2^{\frac{2}{3}+\frac{1}{2}-\frac{7}{6}}$

$= 2^{\frac{4+3-7}{6}}$

$= 2^0$

$= 1$

(b) $6^{\frac{1}{2}} \times 36^{\frac{1}{4}}$

$= 6^{\frac{1}{2}} \times (6^2)^{\frac{1}{4}}$

$= 6^{\frac{1}{2}} \times 6^{\frac{1}{2}}$

$= 6^1$

$= 6$

Express in the form a^r

Make the bases the same.

(c) $\sqrt[3]{6} \times \sqrt{6} \times \sqrt[6]{6}$

$= 6^{\frac{1}{3}} \times 6^{\frac{1}{2}} \times 6^{\frac{1}{6}}$

$= 6^{\frac{1}{3}+\frac{1}{2}+\frac{1}{6}}$

$= 6^1$

$= 6$

(d) $\sqrt{2} \div \sqrt[6]{32} \times \sqrt[3]{16}$

$= 2^{\frac{1}{2}} \div 32^{\frac{1}{6}} \times 16^{\frac{1}{3}}$

$= 2^{\frac{1}{2}} \div (2^5)^{\frac{1}{6}} \times (2^4)^{\frac{1}{3}}$

$= 2^{\frac{1}{2}} \div 2^{\frac{5}{6}} \times 2^{\frac{4}{3}}$

$= 2^{\frac{1}{2}-\frac{5}{6}+\frac{4}{3}}$

$= 2^1$

$= 2$

Try

Solve the following problems.

1 Perform the following calculations.

(a) $5^{\frac{1}{2}} \times 5^{\frac{1}{3}} \div 5^{-\frac{1}{6}}$

(b) $4^{\frac{5}{6}} \times 2^{\frac{4}{3}}$

(c) $\sqrt{7} \times \sqrt[3]{7} \times \sqrt[6]{7}$

(d) $\sqrt[4]{27} \times \sqrt{27} \div \sqrt[4]{3}$

2 Let $a > 0$ and $b > 0$ Perform the following calculations.

(a) $\sqrt{a^3} \div \sqrt[3]{a} \div \sqrt[6]{a^5}$

(b) $\sqrt[3]{a} \times (ab)^{\frac{1}{2}} \div \sqrt[6]{\frac{a^2}{b^3}}$

Exercise

Solve the following problems.

1 Perform the following calculations.

(a) $2^{-\frac{1}{2}} \times 2^{\frac{5}{6}} \div 2^{\frac{1}{3}}$

(b) $3^{\frac{3}{2}} \times 3^{\frac{4}{3}} \div 3^{\frac{5}{6}}$

(c) $5^{-\frac{1}{6}} \times 5^{\frac{2}{3}} \div 5^{-\frac{1}{2}}$

(d) $9^{\frac{1}{3}} \times 3^{\frac{4}{3}}$

(e) $125^{\frac{1}{4}} \div 5^{\frac{1}{4}}$

(f) $8^{\frac{1}{2}} \times 64^{\frac{1}{6}} \div 16^{\frac{1}{8}}$

(g) $\sqrt{5} \times \sqrt[3]{5} \times \sqrt[6]{5}$

(h) $\sqrt[3]{3^2} \times \sqrt{3} \div \sqrt[6]{3}$

(i) $\sqrt[3]{2} \times 2^2 \div \sqrt{2^3 \times \sqrt[6]{2}}$

(j) $\sqrt{2} \times \sqrt[6]{2} \div \sqrt[3]{4}$

(k) $\sqrt[3]{16} \times 4 \div \sqrt[3]{2}$

(l) $\sqrt[6]{27} \times \sqrt[8]{81} \div \sqrt[3]{9}$

2 Let $a > 0$ and $b > 0$ Perform the following calculations.

(a) $\sqrt{a} \div \sqrt[3]{a^2} \times \sqrt[4]{a^2}$

(b) $\sqrt[6]{a^4 \times} a^2 \div \sqrt{a^3}$

(c) $\sqrt[3]{a^2} \div \sqrt[6]{a} \times \sqrt{a^7}$

(d) $\sqrt[3]{ab}^{2} \div \sqrt[6]{ab^2} \times \sqrt{ab}$

(e) $\sqrt{a\ b^3 \times} \sqrt[3]{ab^2} \div (\sqrt{a^2b^4})^{\frac{1}{3}}$

(f) $(\sqrt[6]{a})^2 \times \sqrt[3]{\frac{a}{b}} \div \sqrt{\frac{b}{a^3}}$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

★ Challenge

Simplify the following expression completely and write your answer as a fully reduced fraction:

$$\left(\frac{\left(\left(\frac{243}{32}\right)^{\frac{-2}{5}} \times \left(\frac{9}{4}\right)^{-1}\right)^{\frac{-1}{2}}}{\left(\frac{125}{64}\right)^{\frac{1}{3}} \times (0.004)^{\frac{-1}{2}}}\right)^{\frac{-3}{2}}$$

Reflect

Consider the following mathematical statement:

For all real numbers $a, \sqrt{a^2} = a$.

Is this statement always, sometimes, or never true? Provide a rigorous mathematical justification or counterexample using both positive and negative values for *a* to defend your assertion.

6-5

MATHEMATICS

Exponential Functions

Chapter 6 Exponents and Logarithms

Concept 2 · Exponential Functions and Their Applications

In modern financial technology, automated algorithms track how asset values scale in real time. When a high-yield digital asset or a micro-investment portfolio compounds interest, earnings accumulate based on the total balance rather than a fixed initial deposit. This compounding effect means the capital expands dramatically over time.

Conversely, when transaction fees or inflation rates erode the purchasing power of capital held in a traditional account, the asset's value decreases by a consistent percentage over set intervals. The portfolio does not lose a fixed number of dollars per day; instead, a consistent fraction of the remaining wealth is chipped away. Mathematically modeling these financial movements requires moving past linear rates of change to analyze operations where the independent variable determines the rate of scaling itself. Guiding Question: "How does the value of the base a in an exponential function $y = a^x$ decide whether money grows through compounding ($a > 1$) or shrinks through decay ($0 < a < 1$)? And how can we find the exact point where growth and decay balance each other?"

Learning Outcomes

Represents the exponential function graphically, and deduces its properties (domain, range, asymptotes, increase and decrease).

Point!

!

$y = a^x (a > 0, a \neq 1)$ is called an exponential function of $\boldsymbol{x}$ with base $\boldsymbol{a}$

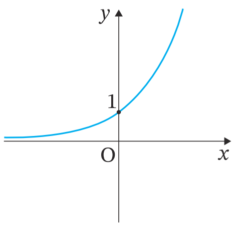
There are 2 types of graphs for $y = a^x (a > 0, a \neq 1)$

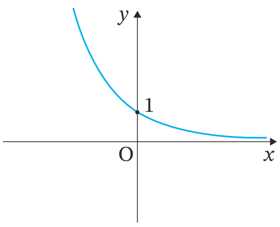
If $a > 1$,

the function is strictly increasing.

If $0 < a < 1$,

the function is strictly decreasing.





To draw the graph of an exponential function, plot the 3 points where $x =$ **− 1, 0, 1** before sketching the curve.

In both cases:

• The domain is the entire set of real numbers, and the range is the entire set of positive numbers.

• The x—axis is a horizontal asymptote.

$y > 0$

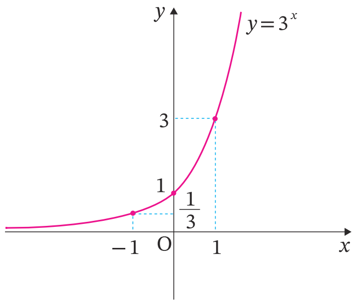
Warm Up

Draw the graphs of the following functions.

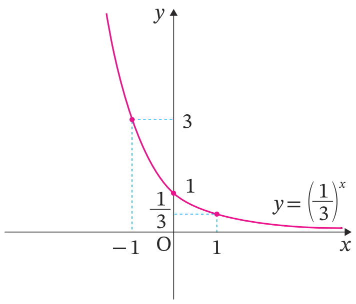
(a) $y = 3^x$ **(b)** $y = \left(\frac{1}{3}\right)^x$

Explanation

(a)



(b)



Try

Draw the graphs of the following functions.

(a) $y = 2^x$ **(b)** $y = \left(\frac{1}{2}\right)^x$

Draw so that it does not intersect the x–axis.

Exercise

Draw the graphs of the following functions.

(a) $y = 4^x$ **(b)** $y = 10^x$

(c) $y = \left(\frac{3}{2}\right)^x$ **(d)** $y = \left(\frac{1}{4}\right)^x$

(e) $y = \left(\frac{1}{10}\right)^x$ **(f)** $y = \left(\frac{2}{3}\right)^x$

Work with a partner

Collaborate with your group members to answer question b, then discuss the answer.

★ Challenge

Draw the graph of the following function:

$$y = \pi^x \text{, where } -1 \leq x \leq 2$$

6-6

MATHEMATICS

Magnitude Relationship of Powers and Roots

Chapter 6 Exponents and Logarithms

Point!

!

Magnitude relationship of powers and roots

If $a > 1$ and $r < s$, then $a^r < a^s$

Example: Since $3 < 5$, $2^3 < 2^5$

If $0 < a < 1$ and $r < s$, then $a^r > a^s$

Example: Since $3 < 5$, $\left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^5$

When the bases are different, **make them the same** .

Learning Outcomes

Compares the magnitudes of powers and roots by applying the laws of exponents.

If the base is greater than 1, the magnitude relationship is the **same** as that of the exponent.

If the base is less than 1, the magnitude relationship is the **reverse** of that of the exponent.

Warm Up

Express the magnitude relationship of the following numbers using inequality signs.

(a) $0.7^2,\ 0.7^{-3},\ 0.7^{-1}$　　**(b)** $\sqrt[5]{8},\ \sqrt[8]{64},\ \sqrt[6]{16}$

Explanation

(a) Since the base 0.7 is less than 1,

regarding the exponent, given that $-3 < -1 < 2$,

$0.7^2 < 0.7^{-1} < 0.7^{-3}$

If $0 <$ base< 1, the inequality sign for the exponents reverses.

(b) First express in the form a^r, and make the bases the same.

$\sqrt[5]{8} = \sqrt[5]{2^3} = 2^{\frac{3}{5}}, \sqrt[8]{64} = \sqrt[8]{2^6} = 2^{\frac{3}{4}}, \sqrt[6]{16} = \sqrt[6]{2^4} = 2^{\frac{2}{3}}$

Since the base 2 is greater than 1,

regarding the exponents, $\frac{3}{5} < \frac{2}{3} < \frac{3}{4}$,

If base > 1, the inequality sign for the exponents stays the same.

$2^{\frac{3}{5}} < 2^{\frac{2}{3}} < 2^{\frac{3}{4}}$

Therefore, $\sqrt[5]{8} < \sqrt[6]{16} < \sqrt[8]{64}$

Try

Express the magnitude relationship of the following numbers using inequality signs.

(a) $5^0,\ 5^{-4},\ 5^{\frac{2}{3}}$ **(b)** $\left(\frac{1}{3}\right)^7,\ \left(\frac{1}{3}\right)^2,\ \left(\frac{1}{3}\right)^{-5}$ **(c)** $\sqrt{3},\ \sqrt[3]{9},\ \sqrt[5]{27}$

Exercise

Express the magnitude relationship of the following numbers using inequality signs.

(a) $2^{0.5},\ 2^5,\ 2^{-2}$ **(b)** $3^2,\ 3^{\frac{1}{5}},\ 3^{-\frac{3}{5}}$ **(c)** $7^{\frac{1}{4}},\ 7^{0.3},\ 7^{-1}$

(d) $0.9^3,\ 0.9^{-3},\ 0.9^2$ **(e)** $0.1^{0.1},\ 0.1^4,\ 0.1^{-3}$ **(f)** $\left(\frac{1}{2}\right)^2,\ \left(\frac{1}{2}\right)^{-3},\ \left(\frac{1}{2}\right)^4$

(g) $\sqrt{2}, \sqrt[3]{16},\ \sqrt[5]{128}$ **(h)** $\sqrt{5},\ \sqrt[3]{25},\ \sqrt[4]{125}$ **(i)** $\sqrt{27},\ \sqrt[3]{81},\ \sqrt[4]{243}$

Work with a partner

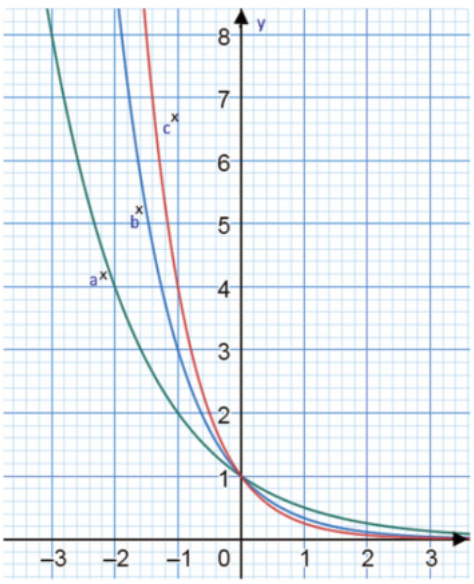
Collaborate with your group members to answer try section, then discuss the answer.

★ Challenge

Express the magnitude relationship of the following three numbers using inequality signs:

$$\text{a)}\ \left(\sqrt{2}\right)^5,\ \left(\sqrt[3]{4}\right)^3,\ \left(\sqrt[4]{8}\right)^2$$

b) Use the opposite graph to compare a, b and c.



6-7
MATHEMATICS

Equations and Inequalities Involving Exponential Functions (1)

Chapter 6 Exponents and Logarithms

Point!

!

Steps to solve equations and inequalities involving exponential functions:

1. Make the **bases** the same using a **number greater than** 1.
2. Create and solve an equation or inequality regarding the **exponents** .

   For inequalities, formulate them after stating that the base is greater than 1

Learning Outcomes

Solves exponential equations and inequalities that reduce to common bases or require algebraic factorization.

It is less error-prone to make the bases the same using a number greater than 1

Warm Up

Solve the following equations and inequalities.

(a) $4^{x-1} = 8^{x+3}$ **(b)** $\left(\frac{1}{3}\right)^{x} = 81$ **(c)** $7^{x} = 7\sqrt{7}$

(d) $2^{2x-1} \geq 1$ **(e)** $\left(\frac{1}{27}\right)^{x} < \left(\frac{1}{3}\right)^{x+4}$ **(f)** $(\sqrt{2})^{x} \geq 8$

Explanation

(a) $4^{x-1} = 8^{x+3}$

$(2^{2})^{x-1} = (2^{3})^{x+3}$

$2^{2(x-1)} = 2^{3(x+3)}$

Therefore, $2(x-1) = 3(x+3)$

Solving this gives $x = -11$

(b) $\left(\frac{1}{3}\right)^{x} = 81$

$3^{-x} = 3^{4}$

Therefore, $-x = 4$

Solving this gives $x = -4$

(c) $7^{x} = 7\sqrt{7}$

$7^{x} = 7 \times 7^{\frac{1}{2}}$

$7^{x} = 7^{1+\frac{1}{2}}$

$7^{x} = 7^{\frac{3}{2}}$

Therefore, $x = \frac{3}{2}$

(d) $2^{2x-1} \geq 1$

$2^{2x-1} \geq 2^{0}$

Since the base 2 is greater than 1,

$2x - 1 \geq 0$

Solving this gives $x \geq \frac{1}{2}$

(e) $\left(\frac{1}{27}\right)^x < \left(\frac{1}{3}\right)^{x+4}$

$\left(\frac{1}{3^3}\right)^x < \left(\frac{1}{3}\right)^{x+4}$

$3^{-3x} < 3^{-(x+4)}$

Since the base 3 is

greater than 1,

$-3x < -(x+4)$

Solving this gives $x > 2$

(f) $(\sqrt{2})^x \geq 8$

$\left(2^{\frac{1}{2}}\right)^x \geq 2^3$

$2^{\frac{1}{2}x} \geq 2^3$

Since the base 2 is

greater than 1,

$\frac{1}{2}x \geq 3$

Solving this gives $x \geq 6$

Try

Solve the following equations and inequalities.

(a) $9^x = 243$

(b) $5^{3x+1} = \frac{1}{125}$

(c) $2^{2-x} = 16\sqrt{2}$

(d) $3^x \leq 9^{x+1}$

(e) $\left(\frac{1}{4}\right)^x > \frac{1}{64}$

(f) $\left(\frac{1}{9}\right)^x > 27$

(g) $\sqrt{3^x} \geq 3^{x+2}$

(h) $\left(\frac{1}{4}\right)^x < \sqrt[3]{2}$

Exercise

Solve the following equations and inequalities.

(a) $25^x = 5^{2-x}$

(b) $27^x = 81$

(c) $2 \times 8^x = 32$

(d) $2^{2x+5} = \frac{1}{64}$

(e) $\left(\frac{1}{4}\right)^{x-1} = 32$

(f) $27^{x+1} = \left(\frac{1}{3}\right)^{5-x}$

(g) $4^x = 2\sqrt{2}$

(h) $3^{x+1} = 27\sqrt{3}$

(i) $25^x = 5\sqrt{5}$

(j) $4^x \leq 2^{x+1}$

(k) $27^x > 9^{1-x}$

(l) $25^x > 125^{x+1}$

(m) $\left(\frac{1}{2}\right)^x < \frac{1}{32}$

(n) $\left(\frac{1}{9}\right)^x \leq \frac{1}{27}$

(o) $\left(\frac{1}{4}\right)^{x-2} < \frac{1}{64}$

(p) $\left(\frac{1}{5}\right)^x \leq 125$

(q) $\left(\frac{1}{27}\right)^{x+1} \geq 81$

(r) $\left(\frac{1}{8}\right)^{2x+3} < 32$

(s) $(\sqrt{5})^x > 25$

(t) $\sqrt{2^x} < 2^{x+3}$

(u) $3^{x-2} \geq \frac{1}{3\sqrt{3}}$

(v) $\left(\frac{1}{9}\right)^x \geq \sqrt[3]{3}$

(w) $\left(\frac{1}{3}\right)^{4x+2} \leq \sqrt{27}$

(x) $\left(\frac{1}{5}\right)^x < \sqrt{5} \times 5^x$

❶ Make the bases the same using a number greater than 1

❷ Create an equation regarding the exponents.

❶ Make the bases the same using a number greater than 1

❷ Create an equation regarding the exponents.

$a^m \times a^n = a^{m+n}$

Common Mistakes

(e)

$\left(\frac{1}{27}\right)^x < \left(\frac{1}{3}\right)^{x+4}$

$\left(\frac{1}{3^3}\right)^x < \left(\frac{1}{3}\right)^{x+4}$

$\left(\frac{1}{3}\right)^{3x} < \left(\frac{1}{3}\right)^{x+4}$

$3x < x + 4$

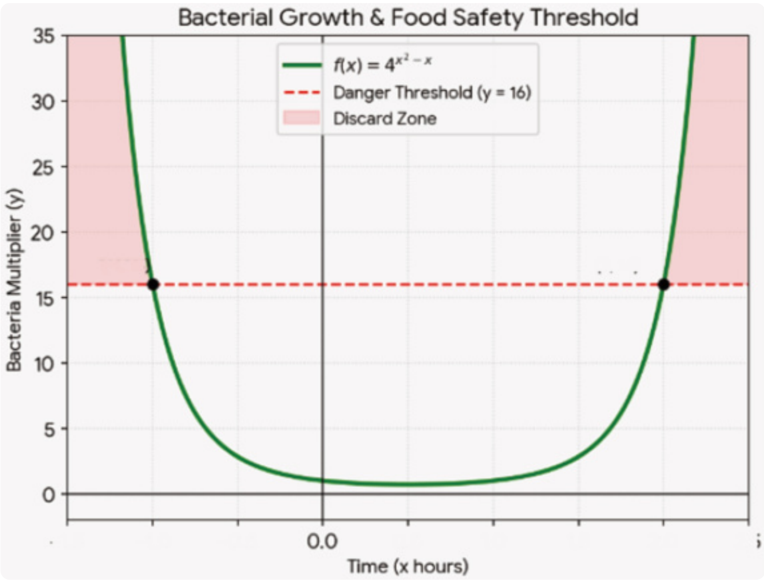
You overlooked that the base is between 0 and 1, which requires flipping the inequality direction.

★ Challenge

An automated food production facility tracks how quickly harmful bacteria multiply when left out of the refrigerator. The initial count starts at 1 million cells per gram. If left unchecked, the bacteria population grows quadratically inside an exponential curve due to a specific nutrient additive. Food safety regulations state the batch must be discarded if the population density reaches or exceeds 16 million cells per gram. The population growth factor follows this mathematical inequality

$$4^{x^2 - x} > 16$$

(where x represents the time in hours past the safe refrigeration window).



How many hours past the safety window must pass before the food becomes unsafe and must be discarded?

Work with a partner

Collaborate with your group members to answer parts (e, f, g, h), then discuss the answer.

6-8

MATHEMATICS

Equations and Inequalities Involving Exponential Functions (2)

Chapter 6 Exponents and Logarithms

Point!

!

Solve equations and inequalities involving a^{2x} and a^x by substituting $\mathbf{a}^x = t$

Then, a^{2x} can be expressed as t^2.

Always specify the condition for t. If $a^x = t$, $t > 0$ must hold.

Finally, check the range of t to solve for x

Learning Outcomes

Solves exponential equations and inequalities that reduce to common bases or require algebraic factorization.

$a^{2x} = (a^x)^2$

Warm Up

Solve the following equations and inequalities.

(a) $4^x - 3 \times 2^{x+1} - 16 = 0$

(b) $3^{2x+1} - 8 \times 3^x - 3 \leq 0$

Explanation

(a) $4^x - 3 \times 2^{x+1} - 16 = 0$

$(2^2)^x - 3 \times (2^x \times 2^1) - 16 = 0$

Make the bases the same using a number greater than 1

$2^{2x} - 6 \times 2^x - 16 = 0$

Substitute $a^x = t$

Let $2^x = t$, then $t > 0$......**(i)**

$t^2 - 6t - 16 = 0$

Solving this gives $t = -2,\ 8$

From **(i)**, $t = 8$

Therefore, $2^x = 8$

$2^x = 2^3$

Therefore, $x = 3$

(b) $3^{2x+1} - 8 \times 3^x - 3 \leq 0$

$(3^{2x} \times 3^1) - 8 \times 3^x - 3 \leq 0$

Make the bases the same using a number greater than 1

$3 \times 3^{2x} - 8 \times 3^x - 3 \leq 0$

Substitute $a^x = t$

Let $3^x = t$, then $t > 0$......**(i)**

$3t^2 - 8t - 3 \leq 0$

Solving this gives $-\frac{1}{3} \leq t \leq 3$...... (ii)

From **(i)** and (ii), $0 < t \leq 3$

Therefore, since $0 < 3^x \leq 3$, we only need to solve for $0 < 3^x$ and $3^x \leq 3$

The solution for $0 < 3^x$ is all real numbers.

$a^x > 0$ holds true for any x

Solving $3^x \leq 3$ gives $x \leq 1$

Therefore, $x \leq 1$

Try

Solve the following equations and inequalities.

(a) $4^x + 4 \times 2^x - 32 = 0$

(b) $9^x - 2 \times 3^{x+1} - 27 = 0$

(c) $4^x - 7 \times 2^x - 8 < 0$

(d) $4^{x+1} - 9 \times 2^x + 2 \geq 0$

Exercise

Solve the following equations and inequalities.

(a) $3^{2x} - 7 \times 3^x - 18 = 0$

(b) $4^x - 5 \times 2^x + 4 = 0$

(c) $9^x + 2 \times 3^x - 15 = 0$

(d) $4^x + 2^{x+1} - 24 = 0$

(e) $4^{x+1} + 7 \times 2^x - 2 = 0$

(f) $3^{2x+1} - 10 \times 3^x + 3 = 0$

(g) $4^x - 3 \times 2^x - 4 \geq 0$

(h) $9^x - 10 \times 3^x + 9 \geq 0$

(i) $4^x - 2^{x+1} - 8 \leq 0$

(j) $3^{2x+1} - 4 \times 3^x + 1 < 0$

(k) $3 \cdot 2^{2x+1} + 2^x - 2 > 0$

(l) $9^{x+1} - 10 \times 3^x + 1 \geq 0$

Work with a partner

Collaborate with your group members to answer parts c, d, then discuss the answer.

★ Challenge

Solve the following inequality completely for all real values of *x* where:

$$\left(\frac{1}{4}\right)^{x^2 - 2x} - 5 \times \left(\frac{1}{2}\right)^{x^2 - 2x} + 4 \leq 0$$

6-9

MATHEMATICS

Maximum and Minimum Values of Exponential Functions

Chapter 6 Exponents and Logarithms

Point!

!

To find the maximum and minimum values of a function involving a^{2x} and a^x, substitute $a^x = t$ and consider it as a quadratic function.

Determine the range of t from the domain and always write it down.

Example: If $-2 \le x \le 3$, letting $3^x = t$ gives $3^{-2} \le t \le 3^3$, so $\frac{1}{9} \le t \le 27$

Learning Outcomes

Deduces the exponential function's maximum and minimum values in applications of growth and decay.

Warm Up

If$-1 \le x \le 3$, find the maximum and minimum values of y for the function $y = 4^x - 2^{x+2}$. Also, find the corresponding values of$x$

Explanation

$y = 4^x - 2^{x+2}$

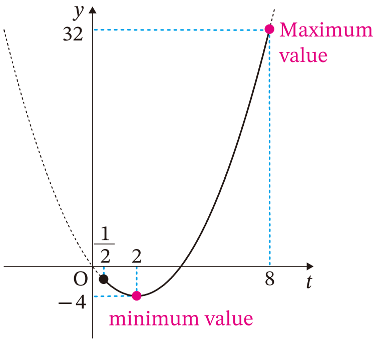
$y = 2^{2x} - 4 \times 2^x$

Let $2^x = t$, then

Since $-1 \le x \le 3,\ 2^{-1} \le t \le 2^3$, so $\frac{1}{2} \le t \le 8$

Determine the range of t from the domain.

$y = t^2 - 4t$



$y = (t-2)^2 - 4$

Therefore, the graph of this function is as shown on the right.

From the graph, it has a maximum value of 32 for $t = 8$,

and a minimum value of-4 for $t = 2$

If $t = 8,\ 2^x = 8$ Solving this gives $x = 3$

If $t = 2,\ 2^x = 2$ Solving this gives $x = 1$

Therefore, the maximum value is 32 for $x = 3$, and the minimum value is-4 for $x = 1$

Try

If $-1 \leq x \leq 2$, find the maximum and minimum values of y for the function $y = -9^x + 2 \times 3^x + 2$ Also, find the corresponding values of x

Exercise

Solve the following problems.

1. If $-3 \leq x \leq 2$, find the maximum and minimum values of y for the function $y = 2 \times 9^x - 12 \times 3^x + 7$ Also, find the corresponding values of x

2. If $-1 \leq x \leq 3$, find the maximum and minimum values of y for the function $y = -4^x + 2^{x+2} - 3$ Also, find the corresponding values of x

3. If $-1 \leq x \leq 2$, find the maximum and minimum values of y for the function $y = 2 \times 4^x - 2^{x+2} + 11$ Also, find the corresponding values of x

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

★ Challenge

An eco-friendly data center utilizes a smart battery bank to manage energy costs during peak operating hours. The heat generated by the server load influences the internal cell resistance.

The hourly **operating cost (in dollars)** of the cooling system follows this function:

$$y = 2 \times (4)^x - 2^{x+2} + 11$$

(where x represents the system's efficiency deviation factor).

Due to mechanical hardware safe limits, the software safety controls restrict the system deviation strictly to the interval $-1 \leq x \leq 2$.

Management needs to calculate the absolute **maximum hourly operating cost** and **minimum hourly operating cost** within this safe operating range to optimize their budget, along with the precise deviation settings x where they occur.

Reflect

Consider the following system of exponential parameters:

$$f(x) = a^x \text{ and } g(x) = \left(\frac{1}{a}\right)^x \text{ where } a > 1$$

If we define a secondary composite function $h(x) = f(x) + g(x)$, analyze the algebraic properties of $h(x)$ to prove whether it is monotonic or non-monotonic across its entire real domain $\mathbb{R}$. Use this behavior to identify its global absolute minimum value.

6-10

MATHEMATICS

Indices and Logarithms

Chapter 6 Exponents and Logarithms

Concept 3 · Introduction to Logarithms and Their Properties

Human sensory perception does not operate on a straight, linear scale. When listening to sound, an audio amplifier increasing the physical power of a signal tenfold does not sound ten times louder to human ears; it registers as a subtle, incremental step up in volume. Similarly, when measuring the acidity of chemical compounds in a laboratory, a solution with a hydrogen ion concentration ten times higher than another is not assigned an intensity value that is ten units larger.

Instead, it shifts by exactly one unit on the standard pH scale. Our biological systems and physical environments compress massive, exponential variances into manageable, ordered scales. To calculate, navigate, and manipulate these compressed dimensions, we must reverse the operation of exponential growth. Guiding Question: If an exponential function maps a known input to a rapidly scaling output ($a^y = x$), how can we structurally define its inverse operation, the logarithm ($\log_a x = y$), to systematically isolate the exponent and convert compounding multiplication into simple addition?

Learning Outcomes

Identifies the concept of the logarithm, and converts expressions between exponential and logarithmic forms.

Point!

!

$\log_a M$ is called the logarithm of M to the base$a$, and M is called the argument of $\log_a M$.log aM

Base Argument

If $a > 0$, $a \neq 1$ and $M > 0$:

$$a^p = M \iff \log_a M = p$$

Example: $2^3 = 8$ → $\log_2 8 = 3$

Example: $\log_3 81 = 4$ → $3^4 = 81$

Swap the last 2 numbers.

Warm Up

Express the following **(a)** in the form $a^p = M$. Express **(b)** in the form $\log_a M = p$

(a) $\log_3 9 = 2$

(b) $2^5 = 32$

Explanation

(a) $\log_3 9 = 2$

$3^2 = 9$

Swap the last 2 numbers.

(b) $2^5 = 32$

$\log_2 32 = 5$

Swap the last 2 numbers.

Try

Express the following **(a)** and **(b)** in the form $a^p = M$. Express **(c)** and **(d)** in the form $\log_a M = p$

(a) $\log_5 125 = 3$

(b) $\log_{\sqrt{2}} (32) = 10$

(c) $8^2 = 64$

(d) $16^{\frac{1}{2}} = 4$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Express the following **(a)** to **(f)** in the form $a^p = M$. Express **(g)** to **(l)** in the form $\log_a M = p$

(a) $\log_{10} 100 = 2$

(b) $\log_3 27 = 3$

(c) $\log_2 128 = 7$

(d) $\log_5 \frac{1}{25} = -2$

(e) $\log_{\frac{1}{2}} \frac{1}{64} = 6$

(f) $\log_{\sqrt{3}} 27 = 6$

(g) $6^3 = 216$

(h) $5^4 = 625$

(i) $9^0 = 1$

(j) $2^{-3} = \frac{1}{8}$

(k) $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

(l) $3^{\frac{1}{2}} = \sqrt{3}$

★ Challenge

If $\text{Log}_5 25 = x$, then evaluate without using a calculator

$$\text{Log}_{\frac{1}{2}}\, x$$

6-11

MATHEMATICS

Values of Logarithms

Chapter 6 Exponents and Logarithms

Point!

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Values of Logarithms

(a) $\log_a 1 = 0$

(b) $\log_a a = 1$

(c) $\log_a a^p = p$

Warm Up

Find the following values.

(a) $\log_{10} 1$ (b) $\log_5 5$ (c) $\log_2 2^8$

(d) $\log_2 16$ (e) $\log_5 \frac{1}{5}$ (f) $\log_7 \sqrt[3]{49}$

Explanation

(a) $\log_{10} 1$

$= 0$

(b) $\log_5 5 = 1$

(c) $\log_2 2^8$

$= 8$

(d) $\log_2 16$

$= \log_2 2^4$

$= 4$

(e) $\log_5 \frac{1}{5}$

$= \log_5 5^{-1}$

$= -1$

(f) $\log_7 \sqrt[3]{49}$

$= \log_7 7^{\frac{2}{3}}$

$= \frac{2}{3}$

Try

Find the following values.

(a) $\log_5 1$ (b) $\log_3 3$ (c) $\log_4 4^4$

(d) $\log_3 27$ (e) $\log_2 \frac{1}{4}$ (f) $log_5 \sqrt{125}$

Learning Outcomes

Calculates logarithmic values.

Since $a^0 = 1$, the value is 0 if the argument is 1

Since $a^1 = a$, the value is 1 if the argument is the same as the base.

From $a^p = M \iff \log_a M = p$, the value is p if the argument is the base raised to the power of p.

The value is 0 if the argument is 1

The value is 1 if the argument and the base are the same.

The value is p if the argument is the base raised to the power of p.

Exercise

Find the following values.

(a) $\log_2 1$ **(b)** $\log_7 7$ **(c)** $\log_3 3^{\frac{1}{2}}$

(d) $\log_3 1$ **(e)** $log_{25}\sqrt{\sqrt{25}}$ **(f)** $\log_{10} 10^{-2}$

(g) $log_2\sqrt{1}$ **(h)** $\log_{\frac{1}{2}} \frac{1}{2}$ **(i)** $\log_4 4^{-\frac{3}{4}}$

(j) $\log_{10} 100$ **(k)** $\log_5 125$ **(l)** $\log_2 64$

(m) $\log_3 \frac{1}{9}$ **(n)** $\log_3 \frac{1}{27}$ **(o)** $\log_5 \frac{1}{125}$

(p) $\log_2 \sqrt{32}$ **(q)** $\log_3 \sqrt[4]{27}$ **(r)** $\log_2 \frac{1}{\sqrt[5]{4}}$

★ Challenge

Put in the simplest form:

$$\text{Log}_{\left(a^3\right)} \left(a^2\right)$$

Express the argument into a power of the base.

The value is p if the argument is the base raised to the power of p.

Express the argument into a power of the base.

The value is p if the argument is the base raised to the power of p.

Express the argument into a power of the base.

The value is p if the argument is the base raised to the power of p.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

6-12 Addition and Subtraction of Logarithms

MATHEMATICS

Chapter 6 Exponents and Logarithms

Point!

! Addition and subtraction of logarithms

• For $M > 0, N > 0$ and let k be a real number. Use the following properties.

(i) $k\log_a M = \log_a M^k$

(ii) If the bases are the same, $\log_a M + \log_a N = \log_a MN$.

$\log_a M - \log_a N = \log_a \dfrac{M}{N}$

Example: $\log_6 2 + \log_6 3$

Example: $\log_2 30 - \log_2 3 - \log_2 5$

$= \log_2 \dfrac{30}{3\times5}$

$= \log_2 2$

$= 1$

• For addition and subtraction of logarithms, calculate after moving the number in front of the log inside as the exponent of the argument.

Example: $2\log_3 12 - 4\log_3 2$

$= \log_3 12^2 - \log_3 2^4$

Learning Outcomes

Calculates logarithmic values and applies the laws of logarithms (product, quotient, powers) in simplifying arithmetic and algebraic operations.

The number in front of the log can be moved inside as the exponent of the argument.

The sum of logarithms corresponds to the multiplication of arguments.

The difference of logarithms corresponds to the division of arguments.

The sum of logarithms corresponds to the multiplication of arguments.

The difference of logarithms corresponds to the division of arguments.

Warm Up

Perform the following calculations.

(a) $4\log_5 3 - 2\log_5 15 - \log_5 45$

(b) $4\log_2\sqrt{2} - \dfrac{1}{2}\log_2 3 + \log_2\dfrac{\sqrt{3}}{2}$

Explanation

(a) $4\log_5 3 - 2\log_5 15 - \log_5 45$

$= \log_5 3^4 - \log_5 15^2 - \log_5 45$

$= \log_5 81 - \log_5 225 - \log_5 45$

$= \log_5 \dfrac{81}{225\cdot 45}$

$= \log_5 \dfrac{1}{125}$

$= \log_5 5^{-3}$

$= -3$

First, move the number in front of the log inside as the exponent of the argument.

Simplify the fraction.

Express the argument as a power of the base.

(b) $4\log_2\sqrt{2} - \frac{1}{2}\log_2 3 + \log_2\frac{\sqrt{3}}{2}$

$$= \log_2(\sqrt{2})^4 - \log_2 3^{\frac{1}{2}} + \log_2\frac{\sqrt{3}}{2}$$

$$= \log_2 4 - \log_2\sqrt{3} + \log_2\frac{\sqrt{3}}{2}$$

$$= \log_2\frac{4\cdot\sqrt{3}}{\sqrt{3}\cdot 2}$$

$$= \log_2 2$$

$$= 1$$

Try

Perform the following calculations.

(a) $\log_6 4 + \log_6 9$

(b) $\log_5 7 - \log_5 35$

(c) $\log_{10} 4 - \log_{10}\frac{1}{25}$

(d) $\log_2 6 + \log_2 12 - 2\log_2 3$

(e) $\frac{1}{2}\log_5 3 + 3\log_5\sqrt{2} - \log_5\sqrt{24}$

(f) $4\log_3\sqrt{3} - \frac{1}{2}\log_3 2 + \log_3\frac{\sqrt{2}}{3}$

Exercise

Perform the following calculations.

(a) $\log_{10} 4 + \log_{10} 25$

(b) $\log_8 2 + \log_8 32$

(c) $\log_6 12 + \log_6 3$

(d) $\log_2 24 - \log_2 3$

(e) $\log_3 7 - \log_3 21$

(f) $\log_5 3 - \log_5 75$

(g) $\log_6 4 - \log_6\frac{1}{9}$

(h) $\log_2 10 - \log_2\frac{5}{2}$

(i) $\log_2 24 - \log_2\frac{3}{4}$

(j) $\log_2 160 + \log_2 20 - 2\log_2 5$

(k) $\log_3 54 + \log_3 6 - 2\log_3 2$

(l) $2\log_5 12 - \log_5 300 - \log_5 60$

(m) $3\log_3\sqrt{3} + \log_3 2\sqrt{3} - \log_3 6$

(n) $3\log_5 3 - 4\log_5\sqrt{15} - \log_5 15$

(o) $\frac{1}{2}\log_2 6 - 3\log_2\sqrt[6]{2} - \log_2\sqrt{3}$

(p) $\log_5\frac{6}{25} - 3\log_5\sqrt[3]{6}$

(q) $2\log_5\sqrt{3} - \frac{1}{2}\log_5 6 + \log_5\frac{\sqrt{6}}{3}$

(r) $\log_2 12 + 4\log_2\frac{2}{3} + 6\log_2\sqrt{3}$

Work with a partner

Collaborate with your group members to answer parts e, f, then discuss the answer.

★ Challenge

Simplify the following fractional logarithmic expression completely to a single reduced fraction:

$$\frac{\text{Log}_5\ 75 + 3\ \text{Log}_5\ \sqrt[3]{2} - 2\ \text{Log}_5\ \sqrt{6}}{\text{Log}_2\ 12 + 4\text{Log}_2\ \frac{2}{3} + 6\ \text{Log}_2\ \sqrt[3]{2}}$$

6-13

MATHEMATICS

Base Conversion (1)

Chapter 6 Exponents and Logarithms

Point!

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Base conversion formula: For $a,\ b,\ c$ being positive numbers, with $a\ \neq\ 1$ and $c\ \neq\ 1$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

In problems finding the value of a logarithm, if the argument cannot be expressed as a power of the base, change the base.

If the base or argument is a power of 2 (such as 4, 8, 16, ...), convert to base **2**

If the base or argument is a power of 3 (such as 9, 27, 81, 243, ...), convert to base **3**

If the base or argument is a power of 5 (such as 25, 125, ...), convert to base **5**

Learning Outcomes

Employs the change-of-base rule to solve logarithmic equations with different bases and to facilitate numerical calculations in mathematical modeling.

Warm Up

Find the following values.

(a) $\log_{16}64$ **(b)** $log_{\sqrt{5}}125$

Explanation

(a) $\log_{16}64$

Since they are 16 and 64, change the base to 2

$$= \frac{\log_2 64}{\log_2 16}$$

$$= \frac{\log_2 2^6}{\log_2 2^4}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

(b) $log_{\sqrt{5}}125$

Since they are 5 and 125, change the base to 5

$$= \frac{\log_5 125}{\log_5 \sqrt{5}}$$

$$= \frac{Log_5\,(5)^3}{Log_5\,(5)^{\frac{1}{2}}}$$

Multiply the numerator and denominator by 2

$$= \frac{3}{\frac{1}{2}}$$

$$= 6$$

Try

Find the following values.

(a) $\log_4 8$ **(b)** $\log_9 3$ **(c)** $\log_{\frac{1}{2}} 16$ **(d)** $\log_3 \sqrt{3}$ **(e)** $\log_4 \sqrt[3]{2}$

Exercise

Find the following values.

(a) $\log_8 4$ **(b)** $\log_4 64$ **(c)** $\log_8 32$ **(d)** $\log_{27} 3$ **(e)** $\log_{125} 5$

(f) $\log_{25} 125$ **(g)** $\log_{\frac{1}{4}} 32$ **(h)** $\log_{\frac{1}{3}} 243$ **(i)** $\log_{\frac{1}{5}} 125$ **(j)** $\log_{\sqrt{2}} 16$

(k) $\log_{\sqrt{3}} 81$ **(l)** $\log_{\sqrt{3}} \frac{1}{27}$ **(m)** $\log_9 \sqrt{3}$ **(n)** $\log_{\frac{1}{4}} \sqrt{2}$ **(o)** $\log_{\frac{1}{5}} \sqrt[3]{5}$

★ Challenge

Find the value of

$$\mathrm{Log}_{\frac{9\sqrt{3}}{4\sqrt{2}}} \left(\frac{27}{8}\right)$$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

6-14 Base Conversion (2)

MATHEMATICS

Chapter 6 Exponents and Logarithms

Point!

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For calculations of logarithms with different bases, first make them the same .

If the argument can be expressed as a power, apply the property $\log_a M^k$ = $k\log_a M$.

$\log_a b$ can be treated as a single variable.

Example: $\log_2 3 + 2\log_2 3 = 3\log_2 3$

Example: $\frac{\log_5 3}{2} \times \frac{4}{3\log_5 3} = \frac{2}{3}$

Learning Outcomes

Expresses logarithms in terms of other logarithms using algebraic properties and the laws of change of base.

a+2a = 3a

$\frac{a}{2} \times \frac{4}{3a} = \frac{2}{3}$

Warm Up

Solve the following problems.

(a) $\log_2 24 - \log_4 36$ **(b)** $\log_9 16 \cdot \log_8 3$ **(c)** $(\log_2 5 + \log_8 5)(\log_5 2 + \log_{25} 2)$

Explanation

(a) $\log_2 24 - \log_4 36$

$$= \log_2 24 - \frac{\log_2 36}{\log_2 4}$$

$$= \log_2 24 - \frac{\log_2 6^2}{\log_2 2^2}$$

$$= \log_2 24 - \frac{2\log_2 6}{2}$$

$$= \log_2 24 - \log_2 6$$

$$= \log_2 \frac{24}{6}$$

$$= \log_2 4$$

$$= \log_2 2^2$$

$$= 2$$

(b) $\log_9 16 \cdot \log_8 3$

$$= \frac{\log_2 16}{\log_2 9} \cdot \frac{\log_2 3}{\log_2 8}$$

$$= \frac{\log_2 2^4}{\log_2 3^2} \cdot \frac{\log_2 3}{\log_2 2^3}$$

$$= \frac{4}{2\log_2 3} \cdot \frac{\log_2 3}{3}$$

$$= \frac{2}{3}$$

(c) $(\log_2 5 + \log_8 5)(\log_5 2 + \log_{25} 2)$

$$= \left(\log_2 5 + \frac{\log_2 5}{\log_2 8}\right)\left(\frac{\log_2 2}{\log_2 5} + \frac{\log_2 2}{\log_2 25}\right)$$

$$= \left(\log_2 5 + \frac{\log_2 5}{\log_2 2^3}\right)\left(\frac{1}{\log_2 5} + \frac{1}{\log_2 5^2}\right)$$

$$= \left(\log_2 5 + \frac{\log_2 5}{3}\right)\left(\frac{1}{\log_2 5} + \frac{1}{2\log_2 5}\right)$$

$$= \left(\frac{3\log_2 5}{3} + \frac{\log_2 5}{3}\right)\left(\frac{2}{2\log_2 5} + \frac{1}{2\log_2 5}\right)$$

$$= \frac{4\log_2 5}{3} \times \frac{3}{2\log_2 5}$$

$$= 2$$

Try

Solve the following problems.

(a) $\log_2 28 - \log_4 49$

(b) $\log_3 5 \cdot \log_5 27$

(c) $\log_{27} 4 \cdot \log_{16} 81$

(d) $\log_2 9 \cdot \log_3 5 \cdot \log_{25} 8$

(e) $(\log_2 3 + \log_4 9)(\log_3 4 + \log_9 2)$

Exercise

Solve the following problems.

(a) $\log_9 4 - \log_3 6$

(b) $\log_2 20 - 3\log_8 5$

(c) $\log_{27} 8 - \log_3 18$

(d) $\log_2 3 \cdot \log_3 4$

(e) $\log_4 3 \cdot \log_3 8$

(f) $\log_3 4 \cdot \log_2 9$

(g) $\log_8 3 \cdot \log_9 4$

(h) $\log_9 16 \cdot \log_4 3$

(i) $\log_4 9 \cdot \log_9 64$

(j) $\log_2 3 \cdot \log_3 5 \cdot \log_5 16$

(k) $\log_2 3 \cdot \log_3 4 \cdot \log_4 2$

(l) $\log_3 16 \cdot \log_2 5 \cdot \log_{125} 9$

(m) $(\log_2 9 + \log_8 3)(\log_3 2 + \log_9 4)$

(n) $(\log_3 4 + \log_9 16)(\log_4 9 + \log_{16} 3)$

(o) $(\log_5 2 + \log_{25} 2)(\log_2 5 + \log_4 125)$

Make the bases the same.

$\log_a M^k = k\log_a M$

Make the bases 2 (you can also make them 3).

Make the bases 2 (you can also make them 5).

Simplify inside the parentheses by finding a common denominator.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

★ Challenge

Simplify the following product of logarithms completely for $x > 1$

$$\mathrm{Log}_x\left(\sqrt{y}\right) \times \mathrm{Log}_{y^2}\left(z^3\right) \times \mathrm{Log}_{\sqrt{z}}\left(x^4\right)$$

Reflect

Consider the following two logarithmic functions:

$$f(x) = \log_4\left(x^2\right) \qquad \text{and} \qquad g(x) = 2\log_4 x$$

Are these two functions identical across the entire coordinate plane? Analyze their structural domains and properties to prove whether their graphs match exactly or diverge over specific real number intervals.

6-15

MATHEMATICS

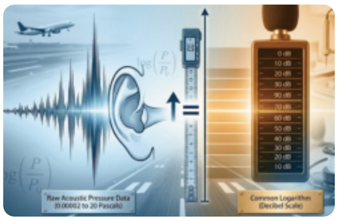
Expressing a Logarithm in Terms of Other Logarithms

Chapter 6 Exponents and Logarithms

Learning Outcomes

Expresses logarithms in terms of other logarithms using algebraic properties and the laws of change of base.

Concept 4 · Applications of Logarithms



Environmental monitoring and computational diagnostics frequently rely on data sets that span several orders of magnitude. When analyzing the environmental noise pollution surrounding an airport, the acoustic pressure waves hitting a worker's eardrums can vary from 0.00002 Pascals at the threshold of human hearing to more than 20 Pascals near a runway. Storing or evaluating these raw pressure values directly would require computing unwieldy data spans.

Instead, acoustics and software platforms rely on the logarithmic decibel scale to balance the data variance. Moving from abstract algebraic operations to practical tools requires analyzing how logarithmic expressions act as dynamic functions. These functions dictate strict boundary conditions, optimize algorithmic constraints, and isolate exact intervals where complex operational forces cross specific safety or performance thresholds.

Guiding Question: How do the rules for the domain ($x > 0$) and the way logarithmic functions increase or decrease depending on the base affect the possible solutions when we solve logarithmic equations, inequalities, or optimization problems?

Point!

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Problems expressing other logarithms in terms of $\log_{10}2$ and $\log_{10}3$

- Prime factorise the argument and rewrite it as a product or quotient of 2, 3, and 10
- When there is a factor of 5, use $5 = \dfrac{10}{2}$
- Use the following properties:

$\log_a MN = \underline{\log_a M + \log_a N}$ $\quad \log_a \dfrac{M}{N} = \underline{\log_a M - \log_a N}$

$\log_a M^k = \underline{k \log_a M}$

Warm Up

When $\log_{10}2 = a$ and $\log_{10}3 = b$, express the following in terms of a and b.

(a) $\log_{10}24$ **(b)** $\log_{10}\dfrac{4}{27}$ **(c)** $\log_{10}45$ **(d)** $\log_{75}24$

Explanation

(a) $\log_{10}24$

$= \log_{10}(2^3 \times 3)$

$= \log_{10}2^3 + \log_{10}3$

$= 3\log_{10}2 + \log_{10}3$

$= 3a+b$

(b) $\log_{10}\frac{4}{27}$

$= \log_{10}\frac{2^2}{3^3}$

$= \log_{10}2^2 - \log_{10}3^3$

$= 2\log_{10}2 - 3\log_{10}3$

$= 2a-3b$

(c) $\log_{10}45$

$= \log_{10}(3^2 \times 5)$

$= \log_{10}\left(3^2 \times \frac{10}{2}\right)$

$= \log_{10}3^2 + \log_{10}\frac{10}{2}$

$= 2\log_{10}3 + \log_{10}10 - \log_{10}2$

$= 2b + 1-a$

(d) $\log_{75}24$

$$= \frac{\log_{10}24}{\log_{10}75}$$

$$= \frac{\log_{10}(2^3 \times 3)}{\log_{10}\left[3 \times \left(\frac{10}{2}\right)^2\right]}$$

$$= \frac{\log_{10}(2^3 \times 3)}{\log_{10}\left(3 \times \frac{10^2}{2^2}\right)}$$

$$= \frac{3\log_{10}2 + \log_{10}3}{\log_{10}3 + 2\log_{10}10 - 2\log_{10}2}$$

$$= \frac{3a+b}{b+2-2a}$$

Try

Given $\log_{10}2 = a$ and $\log_{10}3 = b$, express the following in terms of a and b.

(a) $\log_{10}72$ **(b)** $\log_{10}\frac{8}{3}$ **(c)** $\log_{10}15$ **(d)** $\log_5 12$

Exercise

Given $\log_{10}2 = a$ and $\log_{10}3 = b$, express the following in terms of a and b.

(a) $\log_{10}18$ **(b)** $\log_{10}12$ **(c)** $\log_{10}54$ **(d)** $\log_{10}\frac{16}{9}$

(e) $\log_{10}\frac{81}{8}$ **(f)** $\log_{10}\frac{1}{12}$ **(g)** $\log_{10}5$ **(h)** $\log_{10}25$

(i) $\log_{10}\frac{1}{5}$ **(j)** $\log_{12}18$ **(k)** $\log_{18}5$ **(l)** $\log_{50}60$

★ Challenge

If $\text{Log}_c 3 = x$, $\text{Log}_c 2 = y$, write in terms of x, y each of the following:

1) $\text{Log}_c 12$
2) $\text{Log}_c \frac{3}{8}$
3) $\text{Log}_c \sqrt{6}$
4) $\text{Log}_2 3$
5) $\text{Log}_{3b} 2bc$

Prime factorise the argument.

$\log_a MN = \log_a M + \log_a N$

$\log_a M^k = k\log_a M$

$\log_a \frac{M}{N} = \log_a M - \log_a N$

$5 = \frac{10}{2}$

Change the base to 10 to substitute a and b.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

6-16

MATHEMATICS

Logarithmic Functions

Chapter 6 Exponents and Logarithms

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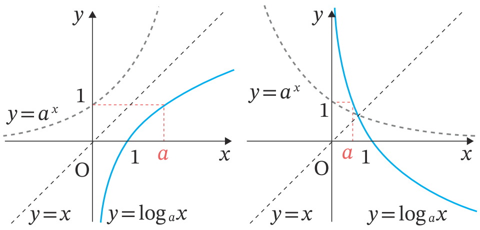
The function $y = \log_a x (a > 0, a \neq 1)$ is called a logarithmic function of x with base a.

There are 2 types of graphs for the logarithmic function $y = \log_a x$.

If $a > 1$, the graph goes up to the right. If $0 < a < 1$, the graph goes down to the right.



In either case:

- The domain is all positive numbers, and the range is all real numbers
- It has the y—axis as an asymptote.
- **It is symmetrical about the line** $y = x$ with the graph of $y = a^x$

Learning Outcomes

Represents the logarithmic function graphically, and deduces its fundamental properties (domain, range, increase and decrease, behavior of the inverse function of the exponential function).

$x > 0$

Warm Up

Solve the following problems.

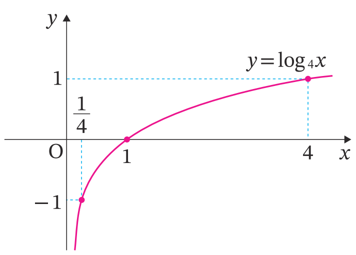
(a) Draw the graph of $y = \log_4 x$. Also, state its positional relationship with $y = 4^x$

(b) Draw the graph of $y = \log_{\frac{1}{4}} x$.

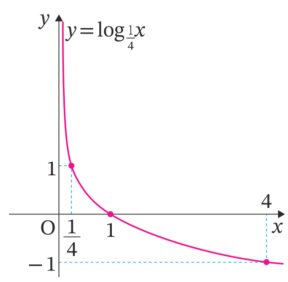
Explanation

It is helpful to plot the points where $y = -1, 0, 1$ and sketch the graph.

(a)



(b)



Draw so that it does not intersect the y—axis.

The graph of $y = \log_4 x$ is symmetrical about the line $y = x$ with the graph of $y = 4^x$

Try

Solve the following problems.

(a) State the positional relationship between the graphs of $y = \log_2 x$ and $y = 2^x$

(b) Draw the graph of $y = \log_2 x$

(c) Draw the graph of $y = \log_{\frac{1}{2}} x$

Exercise

Solve the following problems.

1 Solve the following problems.

(a) State the positional relationship between the graphs of $y = \log_5 x$ and $y = 5^x$

(b) Draw the graph of $y = \log_5 x$

(c) Draw the graph of $y = \log_{\frac{1}{5}} x$

2 Solve the following problems.

(a) State the positional relationship between the graphs of $y = \log_{\frac{1}{3}} x$ and $y = \left(\frac{1}{3}\right)^x$

(b) Draw the graph of $y = \log_3 x$

(c) Draw the graph of $y = \log_{\frac{1}{3}} x$

Work with a partner

Collaborate with your group members to answer c, then discuss the answer.

★ Challenge

Consider the following two transformed functions:

$$f\left(x\right) = \text{Log}_5\left(x-2\right)+1, \quad g\left(x\right) = 5^{(x-1)}+2$$

a) State the exact geometric positional relationship between the graphs of $y = f\left(x\right)$ and $y = g\left(x\right)$.

b) Write the equation of the line of symmetry between the two curves.

6-17
MATHEMATICS

Comparing Logarithms

Chapter 6 Exponents and Logarithms

Point!

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Comparing logarithms

If $a>1$ and $M<N$, then $\log_a M<\log_a N$.

Example: Since $3<5$, $\log_2 3<\log_2 5$

If $0<a<1$ and $M<N$, then $\log_a M>\log_a N$.

Example: Since $3<5$, $\log_{\frac{1}{2}} 3>\log_{\frac{1}{2}} 5$

Steps to compare logarithms:

- If the bases are different, make them the same.
- Express everything in the form $\log_a$

$k\log_a M=\log_a M^k$

$p=\log_a a^p$

Learning Outcomes

Compares different logarithmic values algebraically and graphically using the monotonicity properties of the logarithmic function.

If base a>1, the inequality sign stays the same that of the arguments.

If $0<$base a<1, the inequality sign reverses compared to the arguments.

The number before log becomes the exponent of the argument.

Express integers using log as well.

Warm Up

Express the size relationship of the following numbers using inequality signs.

(a) $\log_2 3,\ \frac{1}{2}\log_2 5,\ 2$

(b) $\log_{\frac{1}{2}} 3,\ \log_{\frac{1}{4}} 5,\ -1$

Explanation

(a) $\frac{1}{2}\log_2 5=\log_2\sqrt{5}$

$2=\log_2 2^2=\log_2 4$

Therefore, compare the sizes of $\log_2 3$, $\log_2\sqrt{5}$, and $\log_2 4$

Comparing the arguments, $\sqrt{5}<3<4$

Since base 2 is greater than 1, $\log_2\sqrt{5}<\log_2 3<\log_2 4$

Thus, $\frac{1}{2}\log_2 5<\log_2 3<2$

(b) $\log_{\frac{1}{4}} 5=\dfrac{\log_{\frac{1}{2}} 5}{2}$

$=\frac{1}{2}\log_{\frac{1}{2}} 5$

$=\log_{\frac{1}{2}}\sqrt{5}$

$-1=\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{-1}$

$=\log_{\frac{1}{2}} 2$

If the logarithm has a coefficient, use $k\log_a M=\log_a M^k$.

Transform integer p into $\log_a a^p$

If the base is greater than 1, it is the same as the size relationship of the arguments.

Answer by converting back to the original form.

Make the base $\frac{1}{2}$

Therefore, compare the sizes of $\log_{\frac{1}{2}} 3$, $\log_{\frac{1}{2}} \sqrt{5}$, and $\log_{\frac{1}{2}} 2$

Comparing the arguments, $2 < \sqrt{5} < 3$

Since base $\frac{1}{2}$ is less than 1, $\log_{\frac{1}{2}} 3 < \log_{\frac{1}{2}} \sqrt{5} < \log_{\frac{1}{2}} 2$

Thus, $\log_{\frac{1}{2}} 3 < \log_{\frac{1}{4}} 5 < -1$

If the base is less than 1, it is the reverse of the size relationship of the arguments.

Answer by converting back to the original form.

Try

Express the size relationship of the following numbers using inequality signs.

(a) $\log_3 5, \log_3 7, 2\log_3 2$

(b) $3\log_{\frac{1}{2}} 3, \ -2\log_{\frac{1}{2}} \frac{1}{5}, \ \frac{5}{2}\log_{\frac{1}{2}} 4$

(c) $\log_2 5, \log_4 3, \ 1$

(d) $\log_{\frac{1}{2}} 3, \log_{\frac{1}{4}} 5, \ -2$

Exercise

Express the size relationship of the following numbers using inequality signs.

(a) $\log_2 0.5, \log_2 3, \log_2 5$

(b) $\log_3 7, 3\log_3 2, \log_3 10$

(c) $\log_2 3, \log_2 8, \ 5$

(d) $\log_{\frac{1}{3}} \frac{1}{2}, \log_{\frac{1}{3}} 3, \ 1$

(e) $\log_{\frac{1}{3}} 4, \ -\log_{\frac{1}{3}} \frac{1}{5}, \ -2$

(f) $-\log_{\frac{1}{2}} 3, \ \frac{1}{2}\log_{\frac{1}{3}} 5, \ -1$

(g) $\log_2 5, \log_4 9, \log_8 64$

(h) $\log_4 5, \log_2 5, \log_8 5$

(i) $\log_3 2, \log_9 7, \ 2$

(j) $\log_{\frac{1}{2}} \sqrt{2}, \log_{\frac{1}{4}} 3, \ -1$

(k) $\log_{\frac{1}{3}} \sqrt{5}, \ -\log_{\frac{1}{9}} \frac{1}{4}, \ -1$

(l) $\frac{1}{2} log_2 5, \ log_{\sqrt{2}} \sqrt{5}, 2$

Work with a partner

Collaborate with your group members to answer c, d, then discuss the answer.

★ Challenge

If $\mathrm{Log}_c 3^{2^{3^2}} = x, \mathrm{Log}_c 9^9 = y$, then write the relation between x, y in the two cases:

i) $0 < c < 1$ **ii)** $c > 1$

6-18
MATHEMATICS

Equations Involving Logarithms

Chapter 6 Exponents and Logarithms

Point!

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For equations involving logarithms, express both sides as logarithms with the same base, and set the arguments equal.

Example: $\log_4 x = 2$

$\log_4 x = \log_4 4^2$

$x = 4^2$

In equations involving logarithms, check the condition (**Argument**) > 0 (Argument Condition).

Example: The argument condition for the logarithm $\log_5(3x-2)$ is $3x-2>0$, so $x > \frac{2}{3}$

Learning Outcomes

Solves logarithmic equations, and systems that require algebraic substitution to change variables, while verifying the validity conditions of the solution (logarithm domain).

Express both sides as logarithms with the same base.

Set the arguments equal.

Warm Up

Solve the following equations.

(a) $\log_3(2x-1) = 2$

(b) $\log_3(x-7) + \log_3(x-5) = 1$

Explanation

(a) Since the argument must be positive, $2x-1>0$, that is, $x > \frac{1}{2}$......**(i)**

Write the argument condition (Solve the inequality).

$\log_3(2x-1) = 2$

$\log_3(2x-1) = \log_3 3^2$

Express both sides as logarithms with the same base.

Therefore, $2x-1 = 3^2$

Set the arguments equal.

$x = 5$

Since this satisfies (i), $x = 5$

(b) Since the arguments must be positive, $x - 7 > 0$ and $x - 5 > 0$, that is, $x > 7 \ldots\ldots$**(i)**

$\log_3(x-7) + \log_3(x-5) = 1$

$\log_3(x-7)(x-5) = \log_3 3$

Therefore, $(x-7)(x-5) = 3$

$x^2 - 12x + 32 = 0$

Solving this, $x = 4,\ 8$

From (i), $x = 4$ is not suitable as a solution.

Thus, $x = 8$

Write the argument condition (Solve the simultaneous inequalities).

Express both sides as logarithms with the same base.

Set the arguments equal.

Try

Solve the following equations.

(a) $\log_5 x = -1$

(b) $\log_3(4x-1) = 3$

(c) $\log_3 x + \log_3(x+2) = 1$

(d) $\log_2(x+1) + \log_2(x-2) = 2$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Solve the following equations.

(a) $\log_3 x = 4$

(b) $\log_4 x = -2$

(c) $\log_3 x = \frac{1}{3}$

(d) $\log_5(x-1) = 2$

(e) $\log_2(x+1) = -1$

(f) $\log_4(x+3) = \frac{1}{2}$

(g) $\log_6 x + \log_6(x-1) = 1$

(h) $\log_2 x + \log_2(x-7) = 3$

(i) $\log_3 x + \log_3(x+8) = 2$

(j) $\log_2(x+5) + \log_2(x-2) = 3$

(k) $\log_3(x+5) + \log_3(x+7) = 1$

(l) $\log_3(x-2) + \log_3(2x-7) = 2$

★ Challenge

Solve the following logarithmic equation completely for all real values of x:

$$\mathrm{Log}_3\left(x+2\right) - \mathrm{Log}_9\left(x-2\right)^2 = 1$$

6-19

MATHEMATICS

Inequalities Involving Logarithms

Chapter 6 Exponents and Logarithms

Point!

!

When solving inequalities involving logarithms, pay attention to the value of the base.

For $\log_a M > \log_a N$, if $a > 1$, $M > N$

If $0 < a < 1$, $M < N$

In inequalities involving logarithms, check the argument condition before finding the solution.

Learning Outcomes

Solves logarithmic inequalities, and systems that require algebraic substitution to change variables, while verifying the validity conditions of the solution (logarithm domain).

Same as the order of the arguments.

Reverse of the order of the arguments.

Warm Up

Solve the following inequalities.

(a) $\log_3 x < 4$

(b) $\log_{\frac{1}{2}}(x+2) + \log_{\frac{1}{2}}(x-5) > -3$

Explanation

(a) Since the argument must be positive, $x > 0$……**(i)**

$\log_3 x < 4$

$\log_3 x < \log_3 3^4$

Since base 3 is greater than 1,

$x < 3^4$

$x < 81$…… (ii)

From (i) and (ii), $0 < x < 81$

Express both sides as logarithms with the same base.

Set up an inequality for the arguments. If the base is greater than 1, the inequality sign remains the same.

The range satisfying the argument condition is the solution.

(b) Since the arguments must be positive, $x + 2 > 0$ and $x - 5 > 0$

That is, $x > 5$……**(i)**

$\log_{\frac{1}{2}}(x+2) + \log_{\frac{1}{2}}(x-5) > -3$

$\log_{\frac{1}{2}}(x+2)(x-5) > \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{-3}$

Since base $\frac{1}{2}$ is less than 1,

$(x+2)(x-5) < \left(\frac{1}{2}\right)^{-3}$

$x^2 - 3x - 18 < 0$

Solving this, $-3 < x < 6$…… (ii)

From (i) and (ii), $5 < x < 6$

Express both sides as logarithms with the same base.

Set up an inequality for the arguments. If the base is less than 1, the inequality sign is reversed.

Try

Solve the following inequalities.

(a) $\log_3 x < 2$

(b) $\log_{\frac{1}{2}} x < 5$

(c) $\log_2 (x-3) < 3$

(d) $\log_{\frac{1}{3}} (x-5) > -1$

(e) $\log_2 (x-2) + \log_2 (x-3) < 1$

(f) $\log_2 (x-1)(x-3) \leq 3$

Exercise

Solve the following inequalities.

(a) $\log_{10} x < 3$

(b) $\log_2 x < 5$

(c) $\log_4 x < -3$

(d) $\log_{\frac{1}{3}} x \leq 1$

(e) $\log_{\frac{1}{2}} x \geq 3$

(f) $\log_{\frac{1}{4}} x \geq -2$

(g) $\log_3 (x-2) > 4$

(h) $\log_{10} (x-5) \leq 1$

(i) $\log_5 (x-2) < -1$

(j) $\log_{\frac{1}{2}} (x+1) > -1$

(k) $\log_{\frac{1}{3}} (x+2) \geq -2$

(l) $\log_{\frac{1}{5}} (x-3) \leq -1$

(m) $\log_{\frac{1}{3}} (4-x) < \log_{\frac{1}{3}} 3x$

(n) $2\log_{\frac{1}{2}} x < \log_{\frac{1}{2}} (2x+3)$

(o) $2\log_{0.1} (x-1) < \log_{0.1} (7-x)$

(p) $\log_2 (x+1) + \log_2 (x-2) < \log_2 10$

(q) $\log_6 (x+1) + \log_6 (x+2) \leq 1$

(r) $\log_{\frac{1}{2}} x + \log_{\frac{1}{2}} (x+1) \geq -1$

(s) $\log_3 (x+2) > 1 + \log_3 x$

(t) $\log_4 (2-x) \leq \log_4 (x+1) + 1$

(u) $\log_{\frac{1}{2}} (x-1) \geq \log_{\frac{1}{2}} (x+3) + 1$

(v) $\log_2 x(x-2) < 3$

(w) $\log_{10} (x+2)(x+5) \leq 1$

(x) $\log_{\frac{1}{2}} (x+4)(x-1) > \log_{\frac{1}{2}} 6$

The range satisfying the argument condition is the solution.

Work with a partner

Collaborate with your group members to answer try section parts e, f, then discuss the answer.

6-20
MATHEMATICS

Equations and Inequalities Using Substitution

Chapter 6 Exponents and Logarithms

Point!

!

For complex logarithmic equations and inequalities, consider substituting $\log_a x = t$

Always check the argument condition.

Learning Outcomes

Solves logarithmic equations and inequalities, and systems that require algebraic substitution to change variables, while verifying the validity conditions of the solution (logarithm domain).

Warm Up

Solve the following problems.

(a) Solve the equation $(\log_2 x)^2 - \log_2 x^2 - 8 = 0$

(b) Solve the inequality $5(\log_2 x)^2 - 8\log_2 x^2 + 3 < 0$

(c) Find the value of $8^{\log_2 5}$

Explanation

(a) Since the argument must be positive, $x > 0 \ldots\ldots(\mathbf{i})$

$(\log_2 x)^2 - \log_2 x^2 - 8 = 0$

$(\log_2 x)^2 - 2\log_2 x - 8 = 0$

Let $\log_2 x = t$ (where t is any real number)

$t^2 - 2t - 8 = 0$

Solving this gives, $t = 4,\ -2$

If $t = 4$, $\log_2 x = 4$ Solving this gives $x = 16$

If $t = -2$, $\log_2 x = -2$ Solving this gives $x = \frac{1}{4}$

Since these satisfy $(\mathbf{i})$, $x = \frac{1}{4},\ 16$

(b) Since the argument must be positive, $x > 0 \ldots\ldots(\mathbf{i})$

$5(\log_2 x)^2 - 8\log_2 x^2 + 3 < 0$

$5(\log_2 x)^2 - 16\log_2 x + 3 < 0$

Let $\log_2 x = t$ (where t is any real number)

$5t^2 - 16t + 3 < 0$

Solving this gives, $\frac{1}{5} < t < 3$

Therefore, $\frac{1}{5} < \log_2 x < 3$

$\log_2 2^{\frac{1}{5}} < \log_2 x < \log_2 2^3$

Since base 2 is greater than 1, $\sqrt[5]{2} < x < 8 \ldots\ldots(\text{ii})$

From $(\mathbf{i})$ and (ii), $\sqrt[5]{2} < x < 8$

(c) Let $8^{\log_2 5} = t(t > 0......(\mathbf{i}))$.

Since $a^p = M$ can be expressed as $\log_a M = p, \log_8 t = \log_2 5$

Solving this gives, $t = 125$ This satisfies $(\mathbf{i})$. Thus, $8^{\log_2 5} = 125$

Try

Solve the following problems.

(a) Solve the equation $(\log_2 x)^2 - \log_2 x^2 - 3 = 0$

(b) Solve the inequality $(\log_3 x)^2 - \log_3 x - 2 \leq 0$

(c) Find the value of $10^{\log_{10} 3}$

Exercise

Solve the following problems.

1 Solve the following equations.

(a) $(\log_2 x)^2 - \log_2 x - 6 = 0$

(b) $(\log_3 x)^2 = \log_3 x^2$

(c) $2(\log_3 x)^2 - 5\log_3 x + 2 = 0$

2 Solve the following inequalities.

(a) $(\log_3 x)^2 - \log_3 x - 6 < 0$

(b) $3(\log_3 x)^2 - \frac{4}{3}\log_3 x^3 + 1 \leq 0$

(c) $4(\log_{\frac{1}{2}} x)^2 - 3\log_{\frac{1}{2}} x - 1 \leq 0$

3 Find the following values.

(a) $3^{\log_3 4}$

(b) $16^{\log_4 3}$

(c) $3^{\log_{27} 7}$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

★ Challenge

A seismological research laboratory monitors low-frequency sound waves inside deep tectonic faults to predict earthquake activity. The structural risk rating *R* of a geological fault line depends on the measured wave intensity *x* following a quadratic logarithmic scale:

$$\left(\text{Log}_3 x\right)^2 - \text{Log}_3 x - 6 < 0$$

(where *x* represents the sound wave energy multiplier, which must be strictly positive: $x > 0$). Safety protocols dictate that as long as the risk rating stays below zero (< 0), the fault line is stable. Seismologists need to find the **exact range of wave intensity *x*** that keeps the fault line in this safe zone.

6-21
MATHEMATICS

Maximum and Minimum of Functions Involving Logarithms

Chapter 6 Exponents and Logarithms

Point!

! For functions involving logarithms, consider substituting $\log_a x = t$

After substituting, find the range of t from the domain.

Learning Outcomes

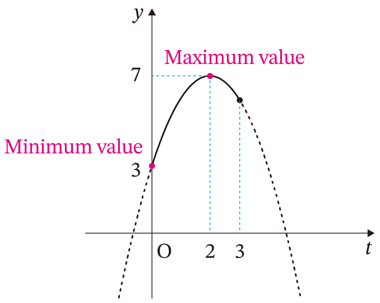
Determines the maximum and minimum values of functions involving logarithmic expressions and deduces their behavior.

Warm Up

Find the maximum and minimum values of the function $y = -(\log_3 x)^2 + 4\log_3 x + 3 (1 \le x \le 27)$. Also, find the corresponding values of x

Explanation

Let $\log_3 x = t$.



Since $1 \le x \le 27$, $\log_3 1 \le \log_3 x \le \log_3 27$

Therefore, $0 \le t \le 3 ......\textbf{(i)}$

$y = -(\log_3 x)^2 + 4\log_3 x + 3$

$y = -t^2 + 4t + 3$

$y = -(t-2)^2 + 7$

Therefore, drawing the graph within the range of $\textbf{(i)}$ results in the figure on the right.

From the graph, the function has a maximum value of 7 for t = 2, and a minimum value of 3 for t = 0

If $t = 2$, $\log_3 x = 2$ Solving this, $x = 9$

If $t = 0$, $\log_3 x = 0$ Solving this, $x = 1$

Therefore, the maximum value is 7 for <u>x = 9</u>, and the minimum value is 3 for <u>x = 1</u>

Try

Find the maximum and minimum values of the function $y = -(\log_2 x)^2 + \log_2 x^2 + 3 (1 \leq x \leq 16)$. Also, find the corresponding values of x

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Solve the following problems.

1 Find the maximum and minimum values of the function $y = (\log_2 x)^2 - \log_2 x^6 + 7 (1 \leq x \leq 16)$. Also, find the corresponding values of x.

2 Find the maximum and minimum values of the function $y = (\log_3 x)^2 - 2\log_3 x^3 + 3 (1 \leq x \leq 27)$. Also, find the corresponding values of x.

3 Find the maximum and minimum values of the function $y = -(\log_2 x)^2 + \log_2 x^4 + 1 (1 \leq x \leq 32)$. Also, find the corresponding values of x.

★ Challenge

A water treatment technician monitors an ecosystem tank. The overall health rating (y) of the aquatic habitat depends on the concentration of a specific dissolved compound (x). The relationship follows this function:

$$y = -\left(\mathrm{Log}_2 x\right)^2 + 2\mathrm{Log}_2 x + 4$$

Due to pump system constraints, the compound concentration is restricted to the interval $1 \leq x \leq 8$

Find the **maximum health rating** and **minimum health rating** of the tank within this range, along with the precise concentration values (x) where they occur.

6-22
MATHEMATICS

Common Logarithms

Chapter 6 Exponents and Logarithms

Point!

!

A logarithm with base 10 is called a common logarithm.

Strategy: To find the number of digits, use the **common logarithm** .

Take the common log of $2^{40} \Longrightarrow \log_{10} 2^{40}$

For an integer N: if $10^{n-1} < N < 10^{n}$, N has exactly n digits.

Example: Since $10^{8} < N < 10^{9}$, N has 9 digits.

For a fraction N: if $10^{-n} < N < 10^{-n+1}$, the first non-zero digit appears at the n-th decimal place.

Example: Since $10^{-4} < N < 10^{-3}$, the first non-zero digit is at the **4th** decimal place.

Learning Outcomes

Employs common logarithms (base 10) in quantitative calculations and applications of mathematical modeling.

Caution: Taking the logarithm changes the value $2^{40} \neq \log_{10} 2^{40}$

Warm Up

Using $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$,solve the following problems.

1 Determine how many digits the integer 6^{30} has.

2 Determine at which decimal place the first non-zero digit appears in $\left(\frac{2}{3}\right)^{50}$

Explanation

1 Taking the common logarithm of 6^{30},

$$\log_{10}6^{30} = 30\log_{10}(2 \times 3)$$
$$= 30(\log_{10}2 + \log_{10}3)$$
$$= 30 \times (0.3010 + 0.4771)$$
$$= 23.343$$

Therefore, since $23 < \log_{10}6^{30} < 24$,

$\log_{10}10^{23} < \log_{10}6^{30} < \log_{10}10^{24}$

Since base 10 is greater than 1, $10^{23} < 6^{30} < 10^{24}$.

Thus, 6^{30} is a 24 –digit integer.

Use the larger absolute value of the exponent as the answer.

2 Use the same approach as finding the number of digits.

Taking the common logarithm of $\left(\frac{2}{3}\right)^{50}$,

$$\log_{10}\left(\frac{2}{3}\right)^{50} = 50\log_{10}\left(2 \times \frac{1}{3}\right)$$

$= 50\left(\log_{10}2 + \log_{10}\frac{1}{3}\right)$

$= 50(\log_{10}2 - \log_{10}3)$

$= 50 \times (0.3010 - 0.4771)$

$= -8.805$

Therefore, since $-9 < \log_{10}\left(\frac{2}{3}\right)^{50} < -8$,

$\log_{10}10^{-9} < \log_{10}\left(\frac{2}{3}\right)^{50} < \log_{10}10^{-8}$

Since base 10 is greater than 1, $10^{-9} < \left(\frac{2}{3}\right)^{50} < 10^{-8}$

Thus, $\left(\frac{2}{3}\right)^{50}$ has its first non-zero digit at the 9th decimal place.

Use the larger absolute value of the exponent as the answer.

Exercise

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Using $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, solve the following problems.

1 Determine how many digits the integer 12^{80} has.

2 Determine at which decimal place the first non-zero digit appears in $\left(\frac{1}{6}\right)^{40}$

Using $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, solve the following problems.

1 Determine how many digits the following integers have.

(a) 2^{30} **(b)** 2^{80} **(c)** 3^{100}

(d) 6^{50} **(e)** 8^{100} **(f)** 15^{70}

2 Determine at which decimal place the first non-zero digit appears for the following numbers.

(a) $\left(\frac{1}{3}\right)^{20}$ **(b)** $\left(\frac{1}{2}\right)^{40}$ **(c)** $\left(\frac{1}{2}\right)^{100}$

(d) $\left(\frac{2}{3}\right)^{100}$ **(e)** $\left(\frac{3}{4}\right)^{80}$ **(f)** $\left(\frac{3}{5}\right)^{30}$

★ Challenge

By using $\text{Log}_{10}2 = 0.3010$ and $\text{Log}_{10}3 = 0.4771$

1. Determine how many digits the integer 12^{50} has.
2. Find the leading (First) digit of 12^{50}

Reflect

Consider the following logarithmic inequality containing a variable base:

$$\log_x(x+2) \le \log_x(4-x)$$

Analyze the system completely. Determine how the variable base *x* forces you to split the problem into separate structural cases, and find the complete real solution set.

7-1

MATHEMATICS

Average Rate of Change

Chapter 7 Differentiation and Integration

Concept 1 · Fundamentals of Calculus (Limits and Derivatives)

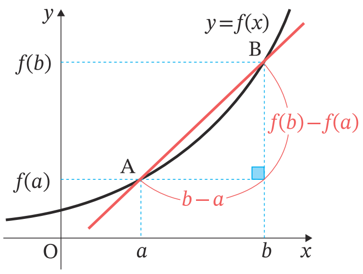
Consider a tumor growing inside a patient's lung or a virus spreading through a population.

Medical professionals monitor these changes continuously, but a critical question arises: how do we calculate the exact rate of growth or decay at one specific, frozen split-second in time? Standard average speed formulas require a time interval, but medicine often requires instantaneous precision. Guiding Question: How can we mathematically evaluate the value of a function or its instantaneous rate of change at a point where the function's traditional formula yields an undefined or mathematically illegal expression like 0 / 0?

Point!

For a function $y = f(x)$, as x changes from a to b,

$$\textbf{Average Rate of Change} = \frac{f(b) - f(a)}{b - a}$$



The average rate of change represents the slope of the line AB.

Warm Up

For the function $f(x) = x^2 + 2x$, find the average rate of change as x changes as follows.

1 As changing from -1 to 3

2 As changing from 3 to $3 + h$

Learning Outcomes

Distinguishes between the geometric and algebraic concept of the average rate of change

Explanation

1 $\dfrac{f(3)-f(-1)}{3-(-1)}$

$= \dfrac{(3^2+2\times 3)-\{(-1)^2+2\times(-1)\}}{4}$

$= \dfrac{15-(-1)}{4}$

$= \dfrac{16}{4}$

$= 4$

2 $\dfrac{f(3+h)-f(3)}{(3+h)-3}$

$= \dfrac{\{(3+h)^2+2(3+h)\}-(3^2+2\times 3)}{\text{h}}$

$= \dfrac{(15+8h+h^2)-15}{h}$

$= \dfrac{8h+h^2}{h}$

$= 8+h$

Try

Find the average rate of change for the following cases.

1 For the function $f(x) = 2x + 3$, as x changes from -2 to 1

2 For the function $f(x) = -x^2 + 2x + 3$, as x changes from 2 to 5

3 For the function $f(x) = x^2 - 2x + 1$, as x changes from -2 to $-2 + h$

Exercise

Find the average rate of change for the following cases.

1 For the function $f(x) = 3x - 1$, as x changes from 1 to 3

2 For the function $f(x) = 2x - 3$, as x changes from 1 to 4

3 For the function $f(x) = 5x + 2$, as x changes from -1 to 1

4 For the function $f(x) = 2x^2$, as x changes from 1 to 3

5 For the function $f(x) = x^2 + 3$, as x changes from 1 to 4

6 For the function $f(x) = -x^2 + 3x - 2$, as x changes from -2 to 0

7 For the function $f(x) = 3x^2$, as x changes from 1 to $1 + h$

8 For the function $f(x) = x^2 + 2x$, as x changes from 2 to $2 + h$

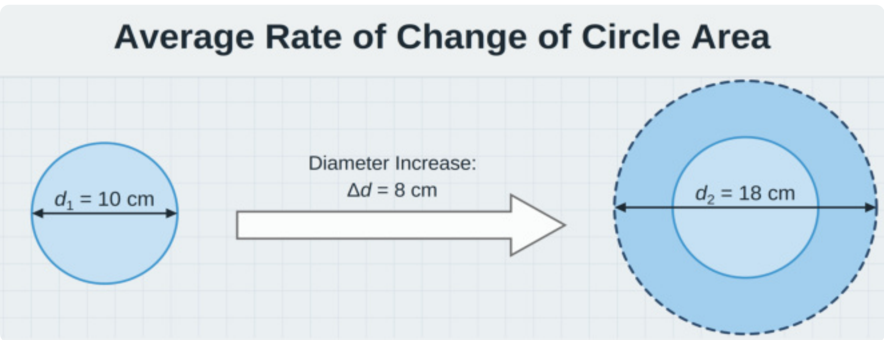
9 For the function $f(x) = -x^2 + x + 1$, as x changes from 3 to $3 + h$

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

★ Challenge

Find the average rate of change of the area of a circle as its diameter length changed from 10 cm to 18 cm.



7-2

MATHEMATICS

Limit Value

Chapter 7 Differentiation and Integration

Point!

!

For a function $f(x)$, as x approaches a, if $f(x)$ approaches a certain constant value α, it is written as:

$\lim_{x \to a} f(x) = \alpha$, and α is called the **limit value** as x approaches a

In the scope of Math II, consider:

$\lim_{x \to a} f(x) = f(a)$

Learning Outcomes

Distinguishes between the geometric and algebraic concept of the average rate of change, and connects it to the value of the limit.

Find it by substituting $x = a$ into $f(x)$

Warm Up

Find the following limit values.

(a) $\lim_{x \to -1} (x^2 - 1)$

(b) $\lim_{h \to 0} \dfrac{6h + 2h^2}{h}$

Explanation

(a) $\lim_{x \to -1} (x^2 - 1)$

$= (-1)^2 - 1$

$= 0$

Find it by substituting $x = -1$

(b) $\lim_{h \to 0} \dfrac{6h + 2h^2}{h}$

$= \lim_{h \to 0} (6 + 2h)$

$= 6 + 2 \times 0$

$= 6$

Substituting directly makes the denominator 0, so simplify the fraction first.

Find it by substituting $h = 0$

Common Mistakes

$\lim_{h \to 0} \dfrac{6h + 2h^2}{h}$

$= \lim_{h \to 0} \dfrac{6 \times 0 + 2 \times 0^2}{0}$

You substituted $\mathrm{h} = 0$ directly from the start, resulting in an undefined form like $\dfrac{0}{0}$

Try

Find the following limit values.

(a) $\lim_{h \to 0}(3 - h)$ **(b)** $\lim_{x \to 3}(x^2 - 4x)$ **(c)** $\lim_{h \to 0}\dfrac{h^3 - 6h^2 + 12h}{h}$

Exercise

Find the following limit values.

(a) $\lim_{h \to 0}(4 + h)$ **(b)** $\lim_{h \to 0}(h - 5)$ **(c)** $\lim_{h \to 0}(3 + 4h)$

(d) $\lim_{x \to 2}(x^2 - x)$ **(e)** $\lim_{x \to 3}(x^2 + 2x)$ **(f)** $\lim_{x \to -2}(x^2 + 3x)$

(g) $\lim_{h \to 0}\dfrac{h^2 - 2h}{h}$ **(h)** $\lim_{h \to 0}\dfrac{h^3 + 2h^2 + 9h}{h}$ **(i)** $\lim_{h \to 0}\dfrac{(3 + h)^2 - 3^2}{h}$

★ Challenge

If $\lim_{x \to 3}\dfrac{x^2 - 5x + k}{x - 3} = L$, where L is a real number, then find the value of $L \times k$.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

7-3 MATHEMATICS

Definition of Differential Coefficient

Chapter 7 Differentiation and Integration

Point!

! Definition of Differential Coefficient $f'(a)$

For a function $f(x)$, the limit value of the average rate of change from $x = a$ to $x = a + h$, which is

$$\frac{f(a+h) - f(a)}{h}$$

as h approaches 0, is called the **differential coefficient** of $f(x)$ at $x = a$, denoted by $\boldsymbol{f'(a)}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Learning Outcomes

Distinguishes between the geometric and algebraic concept of the average rate of change, and connects it to the value of the limit in order to arrive at the definition of the differential coefficient (the first derivative).

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

Warm Up

Find the differential coefficient of $f(x) = 3x^2$ at $x = -2$ using the definition.

Explanation

$$f'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{3(-2+h)^2 - 3 \times (-2)^2}{h}$$

$$= \lim_{h \to 0} \frac{-12h + 3h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-12 + 3h)}{h}$$

$$= \lim_{h \to 0} (-12 + 3h)$$

$$= -12$$

Substitute -2 for a into the definition.

Try

Find the differential coefficients of the following functions at the given values of x using the definition.

(a) $f(x) = 2x^2 [x = -1]$ **(b)** $f(x) = -2x^2 + 3x + 1 [x = 2]$

Exercise

Find the differential coefficients of the following functions at the given values of x using the definition.

(a) $f(x) = x^2 [x = -2]$ **(b)** $f(x) = -x^2 [x = -2]$

(c) $f(x) = -2x^2 [x = -3]$ **(d)** $f(x) = 2x^2 - 3x [x = 2]$

(e) $f(x) = x^2 - 2x + 1 [x = 3]$ **(f)** $f(x) = 2x^2 + x - 1 [x = -2]$

★ Challenge

Let's find the velocity of a dropped stone. In physics, the displacement s (in meters) fallen by an object under gravity after t seconds can be modeled by the function:

$$s(t) = 4.9t^2$$

Find its instantaneous velocity (the differential coefficient of displacement with respect to time) at $t = 10$ sec.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

7-4

MATHEMATICS

Definition of Derivative

Chapter 7 Differentiation and Integration

Point!

!

Definition of Derivative $f'(x)$

For a function $f(x)$, by associating the differential coefficient $f'(a)$ to each value a of x, a new function is obtained.

This new function is called the **derivative** of the original function, denoted by $\boldsymbol{f'(x)}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of finding the derivative $f'(x)$ is called **differentiation**.

Learning Outcomes

Define the derivative using limits, distinguish between average and instantaneous rates of change

This is the formula for the definition of the differential coefficient with a replaced by x

Warm Up

Differentiate the function $f(x) = x^2 + 3x$ using the definition.

Explanation

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\{(x+h)^2 + 3(x+h)\} - (x^2 + 3x)}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2 + 3h}{h} \\ &= \lim_{h \to 0} (2x + h + 3) \\ &= 2x + 3 \end{aligned}$$

Try

Differentiate the following functions using the definition.

(a) $f(x) = 3x^2$ **(b)** $f(x) = x^2 + 2x$ **(c)** $f(x) = 2x^3 + 3x - 1$

Exercise

Differentiate the following functions using the definition.

(a) $f(x) = x^2$ **(b)** $f(x) = 2x^2$ **(c)** $f(x) = -3x^2$

(d) $f(x) = x^2 - 4x$ **(e)** $f(x) = x^2 + 3x + 4$ **(f)** $f(x) = -2x^2 - 3x + 2$

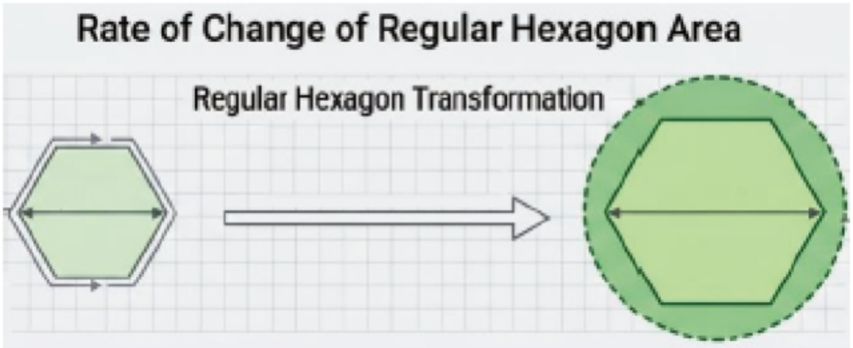
(g) $f(x) = x^3$ **(h)** $f(x) = 2x^3 + x$ **(i)** $f(x) = -x^3 + 2x$

Work with a partner

Collaborate with your group members to answer part c, then discuss the answer.

★ Challenge

A regular hexagon lamina its side length (x) increases regularly, write the rule of its area in terms of (x), then differentiate the function of its area using the definition.



Reflect

A function $f(x)$ is defined such that $f(x) = 5$ for all values of x except $x = 2$, where the function is undefined. Does the limit $\lim_{x \to 2} f(x)$ exist? If so, what is it? Explain why a function does not need to exist at a point for its limit to exist.

7-5

MATHEMATICS

Differentiation using Formulas

Chapter 7 Differentiation and Integration

Biosphere researchers monitoring an endangered wildlife reserve track the population of a species over time. Knowing the total population at the end of the year isn't enough to save them; conservationists must pinpoint the exact rate at which the population is growing or shrinking at any specific week to determine if a sudden ecological threat is active.

In physics and engineering, finding these rapid shifts requires tracking the slope of a curve at a single point. This lesson moves from basic limit calculations to fast, precise algebraic formulas that allow us to reconstruct whole functions and map the exact trajectories of these changing systems. Guiding Question How can we use the derivative of a function at a specific point (x_1, y_1) to construct the linear equation of a geometric line that grazes the curve perfectly at that single point, and how can we reverse this rate of change to determine the original function itself?

Learning Outcomes

Derives the basic rules of differentiation and applies mathematical formulas to find the derivatives of polynomial functions and composite functions.

Point!

!

The derivative $f'(x)$ of the function $y = f(x)$ is also denoted by y', $\dfrac{dy}{dx}$

$\dfrac{dy}{dx}$ means differentiating y with respect to x

Differentiation using Formulas

Except for problems asking to do so "according to the definition", differentiate using the formula $(x^n)' = nx^{n-1}$

Example:

Subtract 1 from the exponent

$(x^3)' = 3x^2$

Bring the exponent down

Subtract 1 from the exponent (x^0 is 1)

$(x)' = 1$

Bring the exponent down

$(1)' = 0$

Thinking of 1 as x^0, the exponent is 0

Differentiating a constant term gives 0

$(4x^3)' = 4 \times 3x^2$

$= 12x^2$

The coefficient 4 remains as is.

$(2x^3 + 5x + 4)' = 2 \times 3x^2 + 5 \times 1 + 0$

$= 6x^2 + 5$

For polynomials, differentiate each term.

Differentiation with respect to variables other than x

- Use the form $\dfrac{d\bigcirc}{d\bigcirc}$ so it is clear which variable was differentiated with respect to.
- Treat variables other than the one being differentiated as constants.

Example:

Differentiate $S = 5gt^2$ with respect to t. $\dfrac{dS}{dt} = \underline{5g} \times \underline{2t}$

$= 10gt$

The symbol $\dfrac{dS}{dt}$ means to differentiate S with respect to t

The differential coefficient $f'(a)$ can be found by substituting $x = a$ into $f'(x)$

Warm Up

Solve the following problems.

1 Differentiate the following functions.

(a) $y = x^2 - 3x + 2$

(b) $y = (2x + 1)^3$

2 Differentiate $V = \pi r^2 h$ with respect to r

3 For the function $f(x) = x^3 - 3x + 1$, find the value of $f'(-1)$

Explanation

1 **(a)** $y = x^2 - 3x + 2$

$y' = 2x - 3$

(b) $y = (2x + 1)^3$

$y = 8x^3 + 12x^2 + 6x + 1$

$y' = 24x^2 + 24x + 6$

2 $V = \pi r^2 h$

$\dfrac{dV}{dr} = \pi h \times (2r)$

$= 2\pi hr$

3 $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$f'(-1) = 3(-1)^2 - 3$

$= 0$

Treat variables other than r as constants.

Expand first.

First find $f'(x)$

Substitute -1 for x into $f'(x)$

Try

Solve the following problems.

1 Differentiate the following functions.

(a) $y = 3x - 2$

(b) $y = -3x^2 + 8x - 5$

(c) $y = (x + 1)(x + 2)$

(d) $y = 2x^3 - 4x^2 + 6x - 7$

(e) $y = (2x - 3)^3$

(f) $y = \dfrac{4}{3}x^3 + \dfrac{1}{2}x^2 - x + 4$

2 Differentiate the following functions with respect to the variable indicated in brackets [ ].

(a) $S = \pi r^2 [r]$

(b) $h = vt - \dfrac{1}{2}gt^2 [t]$

3 For the function $f(x) = 4x^3 - 3x^2 + 2$, find the following values.

(a) $f'(2)$

(b) $f'(-3)$

Exercise

Solve the following problems.

1 Differentiate the following functions.

(a) $y = 16$

(b) $y = -2x$

(c) $y = -4x - 1$

(d) $y = x^2 - 3x + 5$

(e) $y = 3x^2 - 4x + 1$

(f) $y = -4x^2 - 2x - 1$

(g) $y = (2x + 3)^2$

(h) $y = (x + 1)(x - 1)$

(i) $y = (2x - 1)(x + 4)$

(j) $y = 2x^3 - 3x^2 + 2$

(k) $y = -2x^3 + 7x^2 - 3x + 1$

(l) $y = 3x^3 - 5x^2 + 4x - 1$

(m) $y = (3x + 1)^3$

(n) $y = (x + 1)(x - 1)(3 - x)$

(o) $y = (x + 2)(x + 1)(x - 2)$

(p) $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 3x + 1$

(q) $y = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 2x - 5$

(r) $y = \frac{5}{6}x^3 + \frac{1}{8}x^2 - x + 6$

2 Differentiate the following functions with respect to the variable indicated in brackets [ ].

(a) $S = 4\pi r^2[r]$

(b) $V = \frac{1}{3}\pi r^3[r]$

(c) $V = \frac{4}{3}\pi r^3[r]$

(d) $v = gt[t]$

(e) $y = at^2 - bt[t]$

(f) $s = \frac{1}{2}gt^2 - 3t + 4[t]$

3 For the function $f(x) = x^3 + x^2 - 3$, find the following values.

(a) $f'(3)$

(b) $f'(-5)$

4 For the function $f(x) = -x^3 - 3x^2 + 2$, find the following values.

(a) $f'(2)$

(b) $f'(-1)$

5 For the function $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x - 4$, find the following values.

(a) $f'(2)$

(b) $f'(-3)$

★ Challenge

For the function $f(x) = \sqrt[3]{x^5} + \frac{1}{x^2}$, find the value of $f'(1) + f'(-1)$.

Work with a partner

Collaborate with your group members to answer Q3, then discuss the answer.

7-6

MATHEMATICS

Determination of a Function

Chapter 7 Differentiation and Integration

Point!

Differentiating the quadratic function $f(x) = ax^2 + bx + c$ gives

$f'(x) = 2ax + b$

Learning Outcomes

Determine the original function when given certain differential properties (such as its derivative) or specific point conditions.

Warm Up

Find the quadratic function $f(x)$ that satisfies all the following conditions

$f'(0) = -2,\ f'(2) = 2,\ f(1) = 2$

Explanation

Let the required quadratic function be $f(x) = ax^2 + bx + c$, then:

$f'(x) = 2ax + b$

Since $f'(0) = -2,\ b = -2$......(i)

Since $f'(2) = 2,\ 4a + b = 2$ ......(ii)

Since $f(1) = 2,\ a + b + c = 2$......(iii)

Solving the simultaneous equations (i), (ii), and (iii) gives $a = 1,\ b = -2,\ c = 3$

Therefore, $f(x) = x^2 - 2x + 3$

First, find $f'(x)$ to use $f'(0)$ and $f'(2)$

Substitute 0 for x into $f'(x)$

Substitute 2 for x into $f'(x)$

Substitute 1 for x into $f(x)$

Try

Find the quadratic function $f(x)$ that satisfies all the following conditions.

$f'(0) = 2,\ f'(1) = 4,\ f(2) = 6$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Find the quadratic function $f(x)$ that satisfies all the following conditions.

1. $f'(0) = -4,\ f'(2) = 0,\ f(0) = 8$

2. $f'(0) = 3,\ f'(-1) = 8,\ f(3) = -6$

3. $f'(2) = 6,\ f'(1) = 16,\ f(1) = 7$

4. $f'(1) = -4,\ f'(-1) = 8,\ f(3) = -6$

★ Challenge

Find the quadratic function $f(x)$ that satisfies all the following conditions:

$$f'(2) + f'(4) = 0, \quad f(3) = 0 \text{ and } f(1) = 12$$

7-7
MATHEMATICS

Tangent Line (1)

Chapter 7 Differentiation and Integration

Point!

!

The equation of a line with gradient m passing through the point (x_1, y_1) is

$y - y_1 = m(x - x_1)$

The equation of the tangent line is found from the **coordinates of the point of tangency**.

If the coordinates of the point of tangency of $y = f(x)$ are $(a, f(a))$,

- Gradient of the tangent line → $f'(a)$
- Coordinates of the given point → $(a, f(a))$

Learning Outcomes

Apply the first derivative to determine the tangent line equation at a specific point on a curve and analyze the geometric properties associated with it.

Warm Up

Find the equation of the tangent line to the graph of the function $y = x^2 - 4x + 1$ at the point $(3, -2)$

Explanation

Let $f(x) = x^2 - 4x + 1$

It is convenient to set it as $f(x)$ to find $f'(a)$

$f'(x) = 2x - 4$

Therefore, $f'(3) = 2 \times 3 - 4 = 2$

Thus, the required line has a gradient of 2 and passes through the point $(3, -2)$, so:

$y + 2 = 2(x - 3)$

Rearranging this gives $y = 2x - 8$

Try

Find the equation of the tangent line to the graph of the following functions at the given point.

(a) $y = x^2 + 3\ (-2, 7)$

(b) $y = x^2 - 3x + 2\ (1, 0)$

(c) $y = -x^3 + 2x + 4\ (-2, 8)$

Exercise

Find the equation of the tangent line to the graph of the following functions at the given point.

(a) $y = x^2 - 1\,(2,\ 3)$

(b) $y = x^2 + 2x\,(1,\ 3)$

(c) $y = x^2 - 3x\,(2,\ -2)$

(d) $y = x^2 + 3x - 6\,(-1,\ -8)$

(e) $y = 2x^2 - 4x + 3\,(0,\ 3)$

(f) $y = 2x^2 + 3x - 1\,(1,\ 4)$

(g) $y = x^3 - 3x\,(2,\ 2)$

(h) $y = x^3 - 4x\,(1,\ -3)$

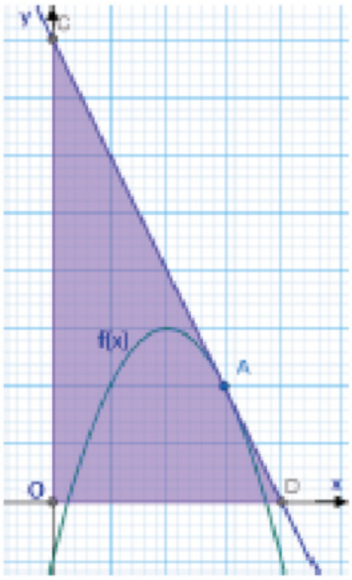
(i) $y = x^3 + 6x^2 + x + 1\,(0,\ 1)$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

★ Challenge

If the tangent to the curve of the function $f(x) = -x^2 + 4x - 1$ at the point whose x coordinate $= 3$ intersects the two axes x, y at the two points D, C respectively, then find the area of triangle ODC.



7-8 Tangent Line (2)

MATHEMATICS

Chapter 7 Differentiation and Integration

Point!

! To find a tangent line and the point of tangency is not given, **let the point of tangency be (a , f (a))**

Learning Outcomes

Apply the first derivative to determine the tangent line equation at a specific point on a curve and analyze the geometric properties associated with it.

Warm Up

Solve the following problems.

1 Find the equation of the tangent line drawn from point C$(0, 2)$ to the graph of the function $y = x^2 - 3x + 6$

2 For the graph of the function $y = x^3 + 4x^2 - 3$, find the equation of the tangent line with a gradient of 3

Explanation

1 Let $f(x) = x^2 - 3x + 6$

It is convenient to set it as $f(x)$

Let the point of tangency of the required tangent line be $(a,\ a^2 - 3a + 6)$

Let the point of tangency be $(a,\ f(a))$

Since $f'(x) = 2x - 3$,

the gradient of the tangent line is $f'(a) = 2a - 3$

Therefore, the required tangent line has a gradient of $2a - 3$ and passes through $(a,\ a^2 - 3a + 6)$, so:

$y - \left(a^2 - 3a + 6\right) = (2a - 3)(x - a)$ ......(i)

Also, since line (i) passes through point C$(0, 2)$:

$2 - \left(a^2 - 3a + 6\right) = (2a - 3)(0 - a)$

Rearranging this gives $a^2 - 4 = 0$

Solving this gives $a = -2,\ 2$

Thus, the equations of the required tangent lines are:

Substituting -2 for a into (i), $y - \{(-2)^2 - 3 \times (-2) + 6\} = \{2 \times (-2) - 3\}\{x - (-2)\}$

Rearranging this gives $y = -7x + 2$

Substituting 2 for a into (i), $y - (2^2 - 3 \times 2 + 6) = (2 \times 2 - 3)(x - 2)$.

Rearranging this gives $y = x + 2y = -7x + 2,\ y = x + 2$

It is convenient to set it as $f(x)$

2 Let $f(x) = x^3 + 4x^2 - 3$

Let the point of tangency be $(a,\ a^3 + 4a^2 - 3)$......(i)

Since $f'(x) = 3x^2 + 8x$,

$f'(a) = 3a^2 + 8a$

Since $f'(a)$ is the gradient of the tangent line, $3a^2 + 8a = 3$

Solving this gives $a = -3,\ \frac{1}{3}$

Substituting these into (i), the points of tangency are $(-3,\ 6)$ or $\left(\frac{1}{3},\ -\frac{68}{27}\right)$

Thus, the equations of the required tangent lines are:

If the point of tangency is$(-3,\ 6)$, $y - 6 = 3\{x - (-3)\}$

Rearranging this gives $y = 3x + 15$

If the point of tangency is $\left(\frac{1}{3},\ -\frac{68}{27}\right)$, $y + \frac{68}{27} = 3\left(x - \frac{1}{3}\right)$

Rearranging this gives $y = 3x - \frac{95}{27}$

$$y = 3x + 15,\ y = 3x - \frac{95}{27}$$

Let the point of tangency be $(a,\ f(a))$

The gradient of the tangent line is 3

Try

Solve the following problems.

1 Find the equation of the tangent line drawn from point C$(1,\ -1)$ to the graph of the function $y = x^2 - x + 3$

2 For the graph of the function $y = x^3 + 2x - 3$, find the equation of the tangent line with a gradient of 5

Exercise

Solve the following problems.

1 Find the equation of the tangent line drawn from point C$(1,\ -2)$ to the graph of the function $y = x^2 + 1$

2 Find the equation of the tangent line drawn from point C$(3, 4)$ to the graph of the function $y = -x^2 + 3x$.

3 Find the equation of the tangent line drawn from point C$(1, 1)$ to the graph of the function $y = -x^2 + x - 3$

4 For the graph of the function $y = x^3 + 3x^2 - 6$, find the equation of the tangent line with a gradient of 9

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

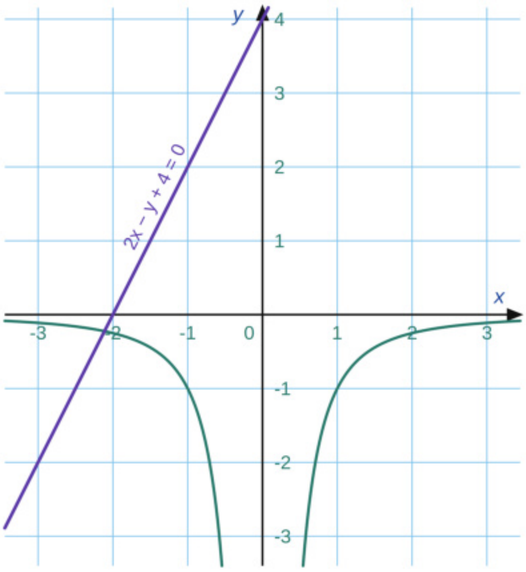
5 For the graph of the function $y = x^3 - x$, find the equation of the tangent line with a gradient of -1

6 For the graph of the function $y = -x^3 + 10x$, find the equation of the tangent line with a gradient of -2

★ Challenge

For the graph of the function $f(x) = -\frac{1}{x^2}$, find the equation of the tangent line which is parallel to the line whose equation:

$$2x - y + 4 = 0$$



Reflect

How does using the tangent line to approximate a curve at a single point help us predict and understand complex real-world patterns, such as changes in weather or shifts in markets, even when those systems seem unpredictable?

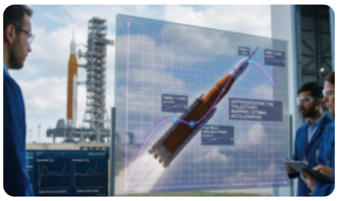
7-9

MATHEMATICS

Graph and Extreme Values of Cubic Functions

Chapter 7 Differentiation and Integration

Concept 3 · Applications of Differentiation: Polynomial Functions



Aerospace engineers designing roller coasters or tracking the launch trajectory of a multi-stage rocket cannot rely on simple straight lines or basic parabolas.

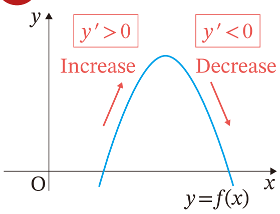
A rocket's fuel consumption, acceleration, and atmospheric drag change drastically as it climbs, producing complex curves with distinct peaks, valleys, and inflection points. To ensure structural safety and fuel efficiency, engineers must know exactly where these peaks occur and determine whether the path intersects dangerous boundary conditions. By analyzing the derivatives of cubic (degree-3) and quartic (degree-4) polynomial functions, we gain the mathematical tools required to map these complex contours, solve higher-degree structural equations, and prove safety thresholds before a single piece of hardware is ever built. Guiding Question How can we use the first derivative of a higher-degree polynomial function to systematically track its turning points, determine the exact number of real roots in its corresponding equation using algebraic discriminants, and construct rigorous proofs for polynomial inequalities?

Learning Outcomes

Analyze the graphs of cubic and quartic functions, identify their critical points, determine intervals of increase and decrease, and classify local and absolute maxima and minima.

Point!

! The increase or decrease of $y = f(x)$ can be found by determining the sign of y'

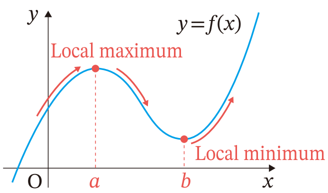


In the range of x where $y' > 0, y = f(x)$ **increases** .

In the range of x where $y' < 0, y = f(x)$ **decreases** .

If the sign of y' changes from positive to negative around $x = a$, $f(a)$ is called a

local maximum .



If the sign of y' changes from negative to positive around $x = b$, $f(b)$ is called a

local minimum .

Local maximums and local minimums are collectively called **extreme values** .

To draw the graph of a cubic function $y = f(x)$, always draw the graph of y' and a sign table.

Warm Up

Find the extreme values of the following functions. Also, draw their graphs.

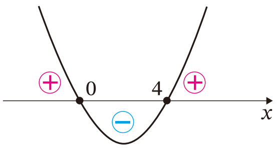
(a) $y = x^3 - 6x^2 + 16$ **(b)** $y = -x^3 + 3x^2 - 5$ **(c)** $y = x^3 - 6x^2 + 12x$

Explanation

(a) Find the derivative. $y = x^3 - 6x^2 + 16 \Longrightarrow y' = 3x^2 - 12x$

Find the values of x where $y' = 0$, and draw the general shape of the graph of y'

Solving $3x^2 - 12x = 0$ gives $x = 0,\ 4$



Read the signs from left to right:
⊕, $\underbrace{0}_{\text{At } x=0}$, ⊖, $\underbrace{0}_{\text{At } x=4}$, ⊕

Draw the graph based on the sign table.

x	……	0	……	4	……
y'	+	0	−	0	+
y	↗	Local maximum	↘	Local minimum	↗

Fill in this row first. Determine the signs from the graph of y'

$y' > 0$ increasing $y' < 0$ decreasing

It has a local maximum at $x = 0$ Substitute 0 for x into $y = x^3 - 6x^2 + 16$ to get the local maximum 16

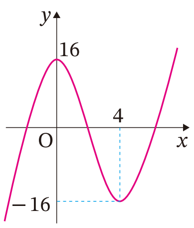
It has a local minimum at $x = 4$ Substitute 4 for x into $y = x^3 - 6x^2 + 16$ to get the local minimum -16

Thus, the sign table is as follows.

x	……	0	……	4	……
y'	+	0	−	0	+
y	↗	Local maximum 16	↘	Local minimum -16	↗

Draw the graph based on the sign table.

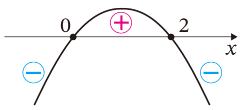
The graph will look like the following figure.



Local maximum 16 at $x = 0$, Local minimum -16 at $x = 4$

(b) $y = -x^3 + 3x^2 - 5$

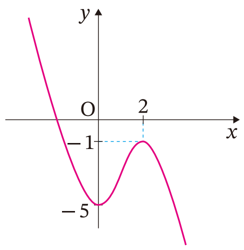
$y' = -3x^2 + 6x$



x	……	0	……	2	……
y'	$-$	0	$+$	0	$-$
y	↘	Local minimum -5	↗	Local maximum -1	↘

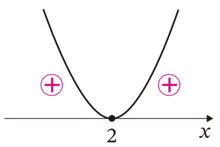
Local minimum -5 at $x = 0,$

Local maximum -1 at $x = 2$



(c) $y = x^3 - 6x^2 + 12x$

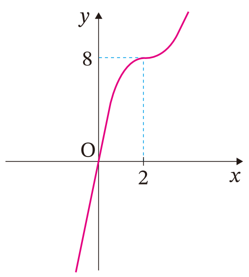
$y' = 3x^2 - 12x + 12$



x	……	2	……
y'	$+$	0	$+$
y	↗	8	↗

Since the sign of y' does not change around $x = 2,$ 8 is not an extreme value.

There are no extreme values.



At $x = 2, y' = 0$ The gradient is 0

Try

Find the extreme values of the following functions. Also, draw their graphs.

(a) $y = x^3 - 3x^2 + 1$ **(b)** $y = -x^3 + 3x + 2$ **(c)** $y = x^3 + 6x^2 + 12x$

Exercise

Find the extreme values of the following functions. Also, draw their graphs.

(a) $y = x^3 - 3x$

(b) $y = x^3 + 3x^2 - 2$

(c) $y = x^3 - 6x^2 + 9x - 4$

(d) $y = -x^3 + 3x$

(e) $y = -2x^3 + 6x^2 - 8$

(f) $y = -x^3 + 6x^2 - 9x + 1$

(g) $y = x^3 - 3x^2 + 3x + 1$

(h) $y = x^3 + 9x^2 + 27x + 25$

(i) $y = -x^3 + 6x^2 - 12x + 1$

Work with a partner

Collaborate with your group members to answer part c, then discuss the answer.

★ Challenge

An engineering team is designing a variable-blade water turbine for a new hydroelectric dam. The efficiency rate of the turbine fluctuates based on the water flow rate. Let x represent the controlled water flow rate entering the chamber (in thousands of liters per second). The net power output efficiency (in megawatts) is modeled by a parametric cubic function:

$$P(x) = x^3 - 3ax^2 + (3a^2 - 3)x + 4$$

where a ($a > 0$) is an adjustable physical parameter corresponding to the pitch angle of the internal turbine blades.

The Engineers' Task is to find the optimal flow rate x that yields the maximum peak power efficiency, and the flow rate that causes the lowest slump in efficiency (expressed in terms of the blade angle parameter a).

7-10
MATHEMATICS

Maximum and Minimum of Cubic Functions

Chapter 7 Differentiation and Integration

Point!

! To find the maximum and minimum values of a cubic function, create a sign table and compare the

extreme values with the **values at** both ends of the domain .

Learning Outcomes

Apply the first derivative test to identify local maxima and minima of cubic functions.

Warm Up

Find the maximum and minimum values of the following function.$y = -x^3 + 12x + 5\,(-3 \le x \le 4)$

Explanation

$y = -x^3 + 12x + 5$

$y' = -3x^2 + 12$

If $y' = 0,\ -3x^2 + 12 = 0$, so $x = \pm 2$

Therefore, the sign table for $-3 \le x \le 4$ is as follows.

x	-3	......	-2	......	2	......	4
y'		$-$	0	$+$	0	$-$	
y	-4	↘	Local minimum -11	↗	Local maximum 21	↘	-11

Consider by comparing the extreme values and the values at both ends.

Thus, this function

has a maximum value of 21 at $x = 2$, and

has a minimum value of -11 at $x = -2,\ 4$

Try

Find the maximum and minimum values of the following functions.

(a) $y = x^3 + 3x^2 - 2\,(-1 \le x \le 2)$ **(b)** $y = -2x^3 + 6x^2 + 3\,(-1 \le x \le 2)$

Exercise

Find the maximum and minimum values of the following functions.

(a) $y = x^3 - 3x^2 + 5\,(-1 \leq x \leq 3)$

(b) $y = x^3 - 3x^2 - 9x + 11\,(-2 \leq x \leq 3)$

(c) $y = 2x^3 - 3x^2 - 12x + 13\,(-3 \leq x \leq 3)$

(d) $y = -x^3 + 3x + 3\,(-2 \leq x \leq 2)$

(e) $y = -x^3 + 3x^2 + 4\,(-2 \leq x \leq 4)$

(f) $y = -2x^3 + 9x^2 - 12x + 3\,(-1 \leq x \leq 4)$

★ Challenge

A marine engineering team is testing an automated deep-sea research submersible in a wave simulation tank. The vessel's depth relative to a steady baseline current layer is tracked over a strict 6-hour testing window.

Let x represent the time in hours relative to the mid-test checkpoint, ranging from 3 hours before checkpoint ($x = -3$) to 3 hours after checkpoint ($x = 3$), so $-3 \leq x \leq 3$. The submersible's vertical position $f(x)$ (in meters relative to the baseline layer) is modeled by:

$$f(x) = x^3 - 6x$$

The Engineers' Task: Find the absolute deepest dive (absolute minimum) and the absolute highest ascent (absolute maximum) achieved by the submersible during the entire 6-hour testing window.

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

7-11

MATHEMATICS

Determination of a Function from Extrema

Chapter 7 Differentiation and Integration

Point!

! If a function $f(x)$ takes an extreme value M at $x = a$, $f(a) = M$ and $\underline{f}'(a) = 0$

Learning Outcomes

Connects the discriminant with the roots of cubic equations in order to solve cubic equations, derive their solutions both geometrically and algebraically

Warm Up

If the function $f(x) = -x^3 + ax + b$ takes a local maximum of 9 at $x = 2$, find the values of constants a and b. Also, find the local minimum.

Explanation

$f(x) = -x^3 + ax + b$

$f'(x) = -3x^2 + a$

Since it has a local maximum of 9 at $x = 2$, $f(2) = 9$ and $f'(2) = 0$

From $f(2) = 9$, $-2^3 + a \times 2 + b = 9$ Rearranging gives $2a + b = 17$......**(i)**

From $f'(2) = 0$, $-3 \times 2^2 + a = 0$ Rearranging gives $a = 12$...... (ii)

Solving the simultaneous equations **(i)** and (ii) gives $a = 12$, $b = -7$

Next, find the local minimum. Substituting 12 for a and -7 for b into $f(x)$ and

$f'(x)$:

x	......	-2	......	2	......
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	Local minimum	↗	Local maximum 9	↘

$f(x) = -x^3 + 12x - 7, f'(x) = -3x^2 + 12$

If $f'(x) = 0$, $-3x^2 + 12 = 0$, so $x = \pm 2$

Thus, the sign table on the right is obtained. Since it has a local minimum at $x = -2$

It takes a local maximum at $x = 2$

$a = 12$, $b = -7$, Local minimum -23 at $x = -2$

Try

Solve the following problems.

1 If the function $f(x) = x^3 - 3x^2 + ax + b$ has a local minimum of 4 at $x = 3$, find the values of constants a and b. Also, find the local maximum.

2 If the function $f(x) = x^3 + ax^2 + bx + c$ has a local maximum of 7 at $x = -1$ and has a local minimum at $x = 3$, find the values of constants $a,\ b,\ c$

Exercise

Solve the following problems.

1 If the function $f(x) = -x^3 + ax^2 + bx + 2$ has a local maximum of 2 at $x = 3$, find the values of constants a and b. Also, find the local minimum.

2 If the function $f(x) = x^3 + ax^2 + bx + 10$ has a local minimum of -70 at $x = 4$, find the values of constants a and b. Also, find the local maximum.

3 If the function $f(x) = x^3 + ax^2 + bx + 1$ has extreme values at $x = -3,\ 1$, find the values of constants a and b

4 If the function $f(x) = x^3 + ax^2 + bx + c$ has a local maximum of 2 at $x = 0$ and a local minimum at $x = 2$, find the values of constants $a,\ b,\ c$

5 If the function $f(x) = 2x^3 + 3ax^2 + bx + c$ has a local maximum of 9 at $x = -1$ and a local minimum at $x = 2$, find the values of constants $a,\ b,\ c$

6 If the function $f(x) = x^3 + ax^2 + bx + c$ has a local maximum of 34 at $x = -1$ and a local minimum at $x = 5$, find the values of constants $a,\ b,\ c$

★ Challenge

An agricultural engineering firm is building a gravity-fed water aqueduct across a desert valley. To prevent flooding, the depth profile of the concrete canal bed must follow a precise cubic trajectory:

$$f(x) = x^3 + ax^2 + bx + c$$

where x represents the horizontal distance (in kilometers) from the primary reservoir gate, and $f(x)$ is the structural elevation (in meters) relative to a baseline datum.

The Design Constraints:

1 **The Safety Crest:** At $x = -2$ kilometers, the canal reaches a local peak altitude of 178 meters to bypass an underground rock formation.

2 **The Valve Trough:** At $x = 4$ kilometers, the canal transitions through a local low turning point to gather silt before the main distribution hub.

The Task:

Determine the exact values of the design constants a, b and c

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

7-12

MATHEMATICS

Cubic Functions and Discriminant (1)

Chapter 7 Differentiation and Integration

Learning Outcomes

Connects the discriminant with the roots of cubic equations in order to solve cubic equations, derive their solutions both geometrically and algebraically

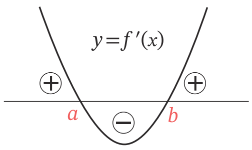
Point!

If there is a condition regarding extreme values of a cubic function $y = f(x)$, check the sign of the discriminant D of

$f'(x) = 0$

- If $y = f(x)$ **has extreme values**, $D > 0$

$f'(x) = 0$ has two distinct real roots.

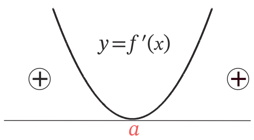


x	……	a	……	b	……
$f'(x)$	+	0	−	0	+
$f(x)$	↗	Local maximum	↘	Local minimum	↗

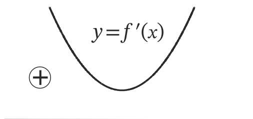
- If $y = f(x)$ **has no extreme values**, $D \le 0$

$f'(x) = 0$ has a repeated real root or no real roots.

$D = 0$ $\quad D < 0$



x	……	a	……
$f'(x)$	+	0	+
$f(x)$	↗		↗



x	……
$f'(x)$	+
$f(x)$	↗

Warm Up

If the function $f(x) = x^3 - 3kx^2 + kx + 2$ has no extreme values, find the range of values for the constant k

Explanation

$f(x) = x^3 - 3kx^2 + kx + 2$

$f'(x) = 3x^2 - 6kx + k$

Since $f(x)$ has no extreme values, the discriminant D of $3x^2 - 6kx + k = 0$ satisfies $D \le 0$, so:

$(-6k)^2 - 4 \times 3 \times k \le 0$

Solving this gives$0 \le k \le \frac{1}{3}$

Try

Solve the following problems.

1 If the function $f(x) = x^3 - 3kx^2 + 3x + 1$ has no extreme values, find the range of values for the constant k

2 If the function $f(x) = x^3 + 2kx^2 - kx + 5$ has extreme values, find the range of values for the constant k

Exercise

Solve the following problems.

1 If the function $f(x) = 2x^3 + 3kx^2 + 6x$ has no extreme values, find the range of values for the constant k

2 If the function $f(x) = x^3 - 2kx^2 + 2kx + 1$ has no extreme values, find the range of values for the constant k

3 If the function $f(x) = x^3 - kx^2 - kx - 2$ has no extreme values, find the range of values for the constant k

4 If the function $f(x) = x^3 + kx^2 + 3x$ has extreme values, find the range of values for the constant k

5 If the function $f(x) = x^3 + 3kx^2 + x - 3$ has extreme values, find the range of values for the constant k

6 If the function $f(x) = x^3 + \frac{3}{2}kx^2 - kx + 1$ has extreme values, find the range of values for the constant k

★ Challenge

An amusement park ride engineer is designing the initial drop section of a new high-speed rollercoaster. To ensure an exciting ride, the track profile must contain both a local peak (crest) and a local drop (valley). The vertical elevation profile of this section is modeled by the cubic function:

$f(x) = x^3 + 3kx^2 + 12kx + 10$

where x represents the horizontal distance (in decameters) from the start of the platform, $f(x)$ is the height (in meters), and k is an adjustable structural reinforcement parameter.

The Engineers' Task:

Determine the exact range of values for the parameter k that guarantees the rollercoaster track will successfully form both a local maximum and a local minimum turning point (extreme values).

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

7-13

MATHEMATICS

Cubic Functions and Discriminant (2)

Chapter 7 Differentiation and Integration

Point!

! If a cubic function $y = f(x)$ is **always monotonically increasing**, for $f'(x) = 0$, $D \leq 0$ and the coefficient of x^2 is positive.

Learning Outcomes

Connects the discriminant with the roots of cubic equations in order to solve cubic equations, derive their solutions both geometrically and algebraically

Warm Up

If the function $f(x) = x^3 - 3kx^2 + (k^2 + 5)x + 3$ is always monotonically increasing, find the range of values for the constant k

Explanation

$f(x) = x^3 - 3kx^2 + (k^2 + 5)x + 3$

$f'(x) = 3x^2 - 6kx + (k^2 + 5)$

Since $f(x)$ is monotonically increasing, the discriminant D of $3x^2 - 6kx + (k^2 + 5) = 0$ satisfies $D \leq 0$, so:

$(-6k)^2 - 4 \times 3 \times (k^2 + 5) \leq 0$

Solving this gives $\underline{-\frac{\sqrt{10}}{2} \leq k \leq \frac{\sqrt{10}}{2}}$

Try

Solve the following problems.

1 If the function $f(x) = x^3 - 3kx^2 + 3kx$ is monotonically increasing, find the range of values for the constant k

2 If the function $f(x) = x^3 + 2kx^2 + (k^2 + 4)x - 1$ is monotonically increasing, find the range of values for the constant k

Exercise

Solve the following problems.

1 If the function $f(x) = x^3 - 3kx^2 + 3x - 4$ is monotonically increasing, find the range of values for the constant k

2 If the function $f(x) = x^3 - x^2 + kx + 2$ is monotonically increasing, find the range of values for the constant k

3 If the function $f(x) = x^3 + kx^2 - 2kx + 5$ is monotonically increasing, find the range of values for the constant k

4 If the function $f(x) = x^3 + kx^2 + (k+6)x + 5$ is monotonically increasing, find the range of values for the constant k

5 If the function $f(x) = x^3 + kx^2 + \left(-k^2 + 12\right)x + 4$ is monotonically increasing, find the range of values for the constant k

6 If the function $f(x) = x^3 + 2kx^2 + \left(-k^2 + 7\right)x - 1$ is monotonically increasing, find the range of values for the constant k

★ Challenge

A green energy corporation is analyzing the performance efficiency index of a smart power grid system. The efficiency of the grid is tracked as the distribution load increases beyond a baseline capacity. Let x represent the distribution load factor, restricted to the operational expansion range $x \in [1, \infty]$. The grid's system stability rating is modeled by the cubic function:

$$f(x) = \frac{1}{3}x^3 - kx^2 + (k+6)x + 10$$

where k is an adjustable voltage regulation constant.
Determine the exact range of values for the constant k such that the stability rating is monotonically increasing for all operational loads in the expansion interval $[1, \infty]$.

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

7-14
MATHEMATICS

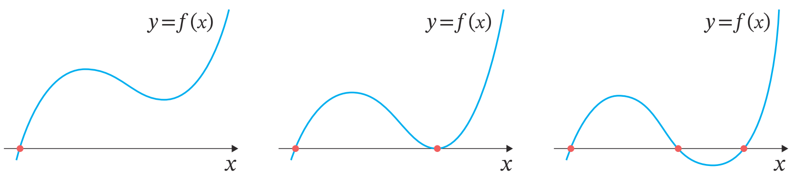
Cubic Functions and Cubic Equations (1)

Chapter 7 Differentiation and Integration

Point!

! The number of real roots of the cubic equation $f(x) = 0$ corresponds to the number of x—intercepts of the graph of the cubic function $y = f(x)$

For problems finding the number of distinct real roots of $f(x) = 0$, consider it by drawing the graph of $y = f(x)$



1 real root   2 real roots   3 real roots

Learning Outcomes

Analyze cubic equations using derivatives to determine critical points, evaluate the increasing and decreasing behavior of the function, and use this analysis to identify the exact number of distinct real roots of the equation.

Warm Up

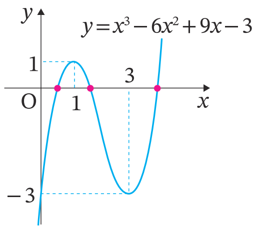
Find the number of distinct real roots for the equation $x^3 - 6x^2 + 9x - 3 = 0$

Explanation

Let $f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9$

If $f'(x) = 0$, $x = 1,\ 3$



Draw the graph of $y = f(x)$

x	......	1	......	3	......
$f'(x)$	+	0	−	0	+
$f(x)$	↗	1	↘	−3	↗

Thus, from the graph, the equation has 3 real roots.

Try

Find the number of distinct real roots for the following equations.

(a) $-x^3 + 12x = 0$ **(b)** $-4x^3 + 12x^2 + 16 = 0$ **(c)** $x^3 - 3x - 2 = 0$

Exercise

Find the number of distinct real roots for the following equations.

(a) $-x^3+2=0$

(b) $x^3+2x^2+x=0$

(c) $4x^3-12x=0$

(d) $x^3-3x^2=0$

(e) $x^3-3x^2+3x+1=0$

(f) $x^3+6x^2-36x=0$

(g) $3x^3+18x^2+12=0$

(h) $x^3-3x^2-9x-5=0$

(i) $2x^3+18x^2+30x-1=0$

★ Challenge

Find the number of distinct real roots for the following equation

$-h^3+12h^2-36h+40=0$

Work with a partner

Collaborate with your group members to answer part c, then discuss the answer.

7-15

MATHEMATICS

Cubic Functions and Cubic Equations (2)

Chapter 7 Differentiation and Integration

Point!

! If there is a parameter (variable letter) in the constant term of the equation, rewrite it as **f(x) = (parameter)**, and consider the number of points of intersection between the graph of $y = f(x)$ and the line $y =$ (parameter).

Learning Outcomes

Analyze cubic equations using derivatives to determine critical points, evaluate the increasing and decreasing behavior of the function, and use this analysis to identify the exact number of distinct real roots of the equation.

Warm Up

If the equation $-x^3 + 3x^2 + 9x + a = 0$ has 3 distinct real roots, find the range of values for the constant a.

Explanation

$$-x^3 + 3x^2 + 9x + a = 0$$

$$-x^3 + 3x^2 + 9x = -a$$

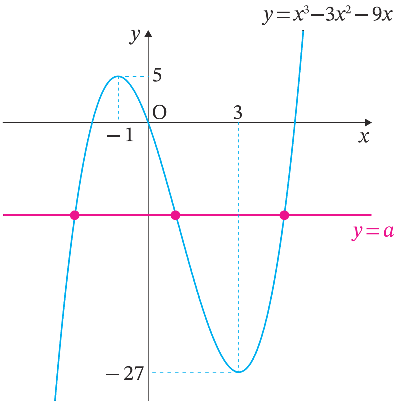
$$x^3 - 3x^2 - 9x = a$$

Rewrite as $f(x) =$ (parameter).

It is easier to think if the parameter is positive, so multiply both sides by -1

The number of roots of this equation is equal to the number of points of intersection between the curve $y = x^3 - 3x^2 - 9x$ and the line $y = a$

Let $f(x) = x^3 - 3x^2 - 9x$



$f'(x) = 3x^2 - 6x - 9$

If $f'(x) = 0$, $x = -1,\ 3$

x	……	-1	……	3	……
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	5	↘	-27	↗

Thus, from the graph,

The range of values for which curve $y = x^3 - 3x^2 - 9x$ and line $y = a$ intersect at exactly three points is:

$$-27 < a < 5$$

Note that if $a = -27, 5$, the number of points of intersection is 2, so they are not included in the range.

Try

Solve the following problems.

1 If the equation $2x^3-3x^2-12x-a=0$ has 3 distinct real roots, find the range of values for the constant a.

2 If the equation $x^3-3x^2-24x+a=0$ has 2 distinct real roots, find the value (s) of the constant a.

Exercise

Solve the following problems.

1 If $2x^3-12x^2+18x+a=0$ has 3 distinct real roots, find the range of values for the constant a.

2 If $-2x^3-3x^2-a=0$ has 3 distinct real roots, find the range of values for the constant a.

3 If $x^3-3x^2-a=0$ has 2 distinct real roots, find the value(s) of the constant a.

4 If $x^3+5x^2+3x-a=0$ has 2 distinct real roots, find the value(s) of the constant a.

★ Challenge

Let a function be defined as:

$f(x)=x^3-6x^2+9x-2a^2+7a$

Find all real values of the constant parameter a such that the equation $f(x)=2$ has exactly 3 distinct real roots.

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

7-16

MATHEMATICS

Proof of Cubic Inequalities

Chapter 7 Differentiation and Integration

Point!

!

To prove LHS $\geq$ RHS, set $f(x) = \text{LHS} - \text{RHS}$, draw a sign table for $y = f(x)$, and show that $(\text{ minimum value }) \geq 0$

Learning Outcomes

Prove polynomial inequalities by constructing auxiliary functions, analyzing their derivatives, and using sign tables to determine minimum values and verify that the inequality holds within a given domain.

Warm Up

Prove that the inequality $x^3 + 2x^2 - 6x + 5 \geq -x^2 + 3x$ holds for $x \geq 0$

Explanation

[Proof]

Let $f(x) = \left(x^3 + 2x^2 - 6x + 5\right) - \left(-x^2 + 3x\right)$

$f(x) = x^3 + 3x^2 - 9x + 5$

$f'(x) = 3x^2 + 6x - 9$

If $f'(x) = 0$, $x = -3,\ 1$

The sign table in the range $x \geq 0$ is as follows.

x	0	……	1	……
$f'(x)$		$-$	0	$+$
$f(x)$	5	$\searrow$	0	$\nearrow$

Thus, in the range $x \geq 0$, the minimum value of $f(x)$ is 0, so:

If $x \geq 0, f(x) = x^3 + 3x^2 - 9x + 5 \geq 0$

Therefore, when $x \geq 0, x^3 + 2x^2 - 6x + 5 \geq -x^2 + 3x$

★ Challenge

Find the largest real constant k such that the inequality

$x^3 - 3x^2 - 3x + k + 3 \leq 0$

holds for all x in the closed interval $[-1, 3]$.

Try

If $x \geq 0$, prove that the inequality $2x^3 + 1 \geq 3x^2$ holds.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Solve the following problems.

1 Prove that the inequality $x^3 + 3x^2 - 9x + 5 \geq 0$ holds for $x \geq 0$

2 Prove that the inequality $x^3 - x^2 + 1 \geq 2x^3 - x$ holds for $x \leq 0$

3 Prove that the inequality $3x^2 + 1 \geq x^3 + 3x$ holds for $x \leq 1$

7-17
MATHEMATICS

Extreme Values and Graphs of Quartic Functions

Chapter 7 Differentiation and Integration

Point!

For the extreme values and graphs of quartic functions, consider them by drawing a sign table just like with cubic functions.

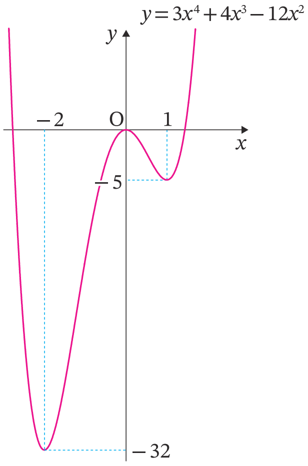
Warm Up

Find the extreme values of the following functions. Also, draw their graphs.

(a) $y = 3x^4 + 4x^3 - 12x^2$ **(b)** $y = -x^4 - 4x^3 - 8$

Explanation

(a) $y = 3x^4 + 4x^3 - 12x^2$



$y' = 12x^3 + 12x^2 - 24x$

$12x^3 + 12x^2 - 24x = 0$

Solving this gives $x = -2,\ 0,\ 1$

x	$\cdots$	-2	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	↘	-32	↗	0	↘	-5	↗

Thus, it has a local maximum of 0 at $x = 0$,

a local minimum of-32 at $x = -2$, and a local minimum of -5 at $x = 1$

Also, the graph is as shown in the figure on the right.

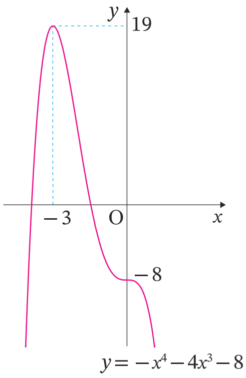
Learning Outcomes

Analyze quartic functions using derivatives to identify local maxima and minima, construct sign tables to determine intervals of increase and decrease, and represent the overall behavior of quartic functions through accurate graphs.

Find x at $y' = 0$

Since y′ is a cubic function, determine its sign by substituting specific values for x, such as x = 3, into y′.

(b) $y = -x^4 - 4x^3 - 8$



$y' = -4x^3 - 12x^2$

$-4x^3 - 12x^2 = 0$

Solving this gives $x = 0, -3$

x	$\cdots$	-3	$\cdots$	0	$\cdots$
y'	$+$	0	$-$	0	$-$
y	$\nearrow$	19	$\searrow$	-8	$\searrow$

Thus, it has a local maximum of 19 at $x = -3$,

and there is no local minimum.

Also, the graph is as shown in the figure on the right.

Find x where $y' = 0$

Since the sign of y' does not change around $x = 0$, it does not have an extreme value at $x = 0$

Try

Find the extreme values of the following functions. Also, draw their graphs.

(a) $y = x^4 - 8x^2 + 5$ **(b)** $y = -x^4 + 6x^2 - 8x$

Exercise

Find the extreme values of the following functions. Also, draw their graphs.

(a) $y = x^4 - 8x^2 + 10$ **(b)** $y = x^4 - 4x^2 + 3$

(c) $y = 3x^4 + 8x^3 - 6x^2 - 24x$ **(d)** $y = x^4 + 2x^3 + 3$

(e) $y = -x^4 + 4x^3 - 9$ **(f)** $y = x^4 + 4x^3 - 16x$

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

Reflect

A quartic function is defined as:

$$f(x) = x^4 - 4x^3 + k$$

For what exact real values of the constant k will this function cross the x-axis exactly twice, creating exactly two distinct real roots? Explain how evaluating the global minimum value allows you to isolate this specific range.

7-18
MATHEMATICS

Indefinite Integral

Chapter 7 Differentiation and Integration

Concept 4 · Integration: Definite Integrals and Applications

In biomedical engineering, tracking the precise delivery of medication into the bloodstream requires measuring the total accumulation of a drug over a given time interval.

If a sensor records a patient's intravenous drug absorption rate as a changing function f(x), finding the cumulative dosage administered between two hours involves summing up infinitely small slices of time and rate. This transition from knowing a changing rate to finding a total accumulated amount is the core power of integration. Guiding Question: How can we mathematically determine the net accumulated change of a physical quantity over a continuous interval when we only know its instantaneous rate of change at any given moment?

Learning Outcomes

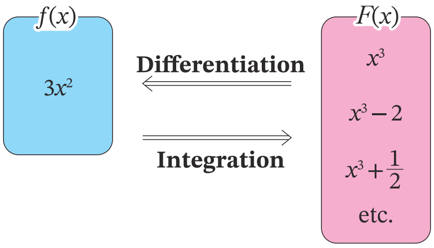
The indefinite integral is recognized as the reverse process of differentiation, and it provides a function when its derivative or the slope of its tangent line is known.

Point!

! For a function $f(x)$, a function $F(x)$ whose derivative is $f(x)$ is called the indefinite integral or primitive function of $f(x)$

Finding the indefinite integral of $f(x)$ is called integrating $f(x)$

Example: If $f(x) = 3x^2$,



As in the example above, there are infinitely many functions $F(x)$ that yield $f(x)$ when differentiated, but since they only differ by a constant term, they can be expressed as $F(x) + C$. These are collectively denoted by $\int \boldsymbol{f}(\boldsymbol{x})\,dx$.($\int$ is read as "integral")

$$\int f(x)\,\boldsymbol{dx} = F(x) + \boldsymbol{C}$$ (where C is the constant of integration)

Always add to the note, "C is the constant of integration." (It will be omitted hereafter in this textbook.)

Indefinite integral of x^n

$$\int \boldsymbol{x}^{n}\boldsymbol{dx} = \frac{1}{\boldsymbol{n}+1}\boldsymbol{x}^{n+1} + \boldsymbol{C}$$

+1 +1

Example: $\int x^3 dx = \frac{1}{4}x^4 + C \quad \int x dx = \frac{1}{2}x^2 + C \quad \int dx = x + C$

+1 +1

1 is omitted and $\int$ 1dx is written as $\int$ dx.
Think of 1 as x^0

If $F'(x) = f(x)$, $G'(x) = g(x)$, and C be the constant of integration, the following equations hold.

$$\int kf(x)\,dx = kF(x) + C \text{ (}k\text{ is a constant)}$$

Example: $\int 3x^3 dx = 3 \times \frac{1}{4}x^4 + C = \frac{3}{4}x^4 + C$

$$\int \{f(x) + g(x)\}dx = F(x) + G(x) + C$$

Example: $\int (x^2 + x)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

$$\int \{f(x) - g(x)\}dx = F(x) - G(x) + C$$

Example: $\int (2x - 3)\,dx = x^2 - 3x + C$

Learn to calculate without writing out the intermediate steps.

Warm Up

Find the following indefinite integrals.

(a) $\int (x^3 - 9x^2 + 3x - 5)\,dx$

(b) $\int (x - 1)^2\,dx$

Explanation

(a) $\int (x^3 - 9x^2 + 3x - 5)\,dx$

$= \frac{1}{4}x^4 - 3x^3 + \frac{3}{2}x^2 - 5x + C$

(b) $\int (x - 1)^2\,dx$

$= \int (x^2 - 2x + 1)\,dx$

$= \frac{1}{3}x^3 - x^2 + x + C$

Expand the expression first.

Try

Find the following indefinite integrals.

(a) $\int (-3)\,dx$

(b) $\int (4x - 5)\,dx$

(c) $\int (3x^2 - 6x + 2)\,dx$

(d) $\int (4x^3 - 2x^2 + x + 1)\,dx$

(e) $\int (2x - 3)^2\,dx$

(f) $\int (x - 1)(x + 2)\,dx$

Exercise

Find the following indefinite integrals.

(a) $\int 2dx$

(b) $\int (-1)\,dx$

(c) $\int \frac{1}{2}dx$

(d) $\int (2x+5)\,dx$

(e) $\int (-3x-2)\,dx$

(f) $\int \left(\frac{1}{2}x+7\right)\,dx$

(g) $\int \left(x^2-5x+1\right)\,dx$

(h) $\int \left(6x^2-6x+2\right)\,dx$

(i) $\int \left(-3x^2+6x+2\right)\,dx$

(j) $\int \left(4x^3+6x^2-10x+2\right)\,dx$

(k) $\int \left(-2x^3-3x^2-4x+3\right)\,dx$

(l) $\int \left(x^3+x^2+x+1\right)\,dx$

(m) $\int (x+1)^2\,dx$

(n) $\int (2x-1)^2\,dx$

(o) $\int (3x-2)^2\,dx$

(p) $\int (3x+2)(3x-2)\,dx$

(q) $\int (3x+2)(3x-3)\,dx$

(r) $\int (3x-1)(x+1)\,dx$

Work with a partner

Collaborate with your group members to answer parts e,f, then discuss the answer.

7-19

MATHEMATICS

Finding a Function from its Derivative

Chapter 7 Differentiation and Integration

Point!

! To find $F(x)$ from its derivative $F'(x)$, **integrate F′ (x) with respect to x** .

$$F(x) = \int F'(x)\,dx$$

Learning Outcomes

The indefinite integral is recognized as the reverse process of differentiation, and it provides a function when its derivative or the slope of its tangent line is known.

Warm Up

Find the function $F(x)$ that satisfies the following conditions. $F'(x) = 3x^2 + 8x - 1$, $F(1) = 7$

Explanation

$$F(x) = \int \left(3x^2 + 8x - 1\right) dx$$

$$= x^3 + 4x^2 - x + C$$

Integrate F′(x).

Since it is an indefinite integral, the constant of integration is undetermined.

Since $F(1) = 7$,

$1^3 + 4 \times 1^2 - 1 + C = 7$

Solving this gives $C = 3$

Therefore, $F(x) = x^3 + 4x^2 - x + 3$

Try

Find the function $F(x)$ that satisfies the following conditions.

(a) $F'(x) = 6x - 2$, $F(0) = 1$

(b) $F'(x) = x^2 - 5x + 3$, $F(2) = 1$

Exercise

Find the function $F(x)$ that satisfies the following conditions.

(a) $F'(x) = 2x - 1$, $F(0) = -3$

(b) $F'(x) = 2x - 2$, $F(1) = 5$

(c) $F'(x) = 4x + 2$, $F(2) = 3$

(d) $F'(x) = 6x^2 + 6x + 2$, $F(0) = 5$

(e) $F'(x) = 3(x-1)(x-2)$, $F(1) = -1$

(f) $F'(x) = 3x^2 - 4x + 3$, $F(2) = 1$

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

7-20

MATHEMATICS

Finding a Function from the Gradient of a Tangent Line

Chapter 7 Differentiation and Integration

Point!

The gradient of the tangent line at each point $(x,\ y)$ on the graph of the function $y = f(x)$ is expressed by **f ′(x)** .

Learning Outcomes

The indefinite integral is recognized as the reverse process of differentiation, and it provides a function when its derivative or the slope of its tangent line is known.

Warm Up

A curve $y = f(x)$ passes through the point $(3, 2)$. If the gradient of the tangent line at each point (x, y) on this curve is given by $3x^2 - 2x + 1$, find $f(x)$

Explanation

Since the gradient of the tangent line at point $(x,\ y)$ is $3x^2 - 2x + 1$,

$f'(x) = 3x^2 - 2x + 1$

Since $f(x) = \int f'(x)\,dx$,

$f(x) = \int (3x^2 - 2x + 1)\,dx$

$= x^3 - x^2 + x + C$

Since the curve $y = f(x)$ passes through the point $(3, 2)$,

$2 = 3^3 - 3^2 + 3 + C$

Solving this gives $C = -19$

Therefore, $f(x) = x^3 - x^2 + x - 19$

Try

A curve $y = f(x)$ passes through the point $(1, 1)$. If the gradient of the tangent line at each point (x, y) on this curve is given by $3x^2 + 2$, find $f(x)$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

Exercise

Solve the following problems.

1 A curve $y = f(x)$ passes through the point $(2, 3)$. If the gradient of the tangent line at each point (x, y) on this curve is $9x^2 - 6x$, find $f(x)$

2 A curve $y = f(x)$ passes through the point $(2,\ -7)$. If the gradient of the tangent line at each point $(x,\ y)$ on this curve is $-6x^2 - 2x + 3$, find $f(x)$

3 A curve $y = f(x)$ passes through the point $(2, 5)$. If the gradient of the tangent line at each point (x, y) on this curve is $12x^2 - 6x - 5$, find $f(x)$

7-21

MATHEMATICS

Definite Integral (1)

Chapter 7 Differentiation and Integration

Point!

! If $F(x)$ is an indefinite integrals of $f(x)$, $\mathrm{F}(\mathbf{b}) - \mathrm{F}(\mathbf{a})$ is called the definite integral of $f(x)$ from a to b, and is denoted by:

$\int_a^b f(x)dx$. Also, $F(b) - F(a)$ is denoted by $[F(x)]_a^b$. That is,

$$\int_a^b f(x)dx = [F(x)]_a^b = \mathrm{F}(\mathbf{b}) - \mathrm{F}(\mathbf{a})$$

In $\int_a^b f(x)dx$, a is called the lower limit of the definite integral, and b is called the upper limit.

Also, finding this definite integral is called integrating $\boldsymbol{f(x)}$ from a to b.

For definite integrals, the following formulas hold.

$$\int_a^a f(x)dx = \underline{0}$$

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b \{f(x) + g(x)\}dx$$

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b \{f(x) - g(x)\}dx$$

Learning Outcomes

Calculate the value of the definite integral using the Fundamental Theorem of Calculus.

If the upper and lower limits are the same, the definite integral is 0

If the upper and lower limits of each definite integral are the same, they can be combined into one definite integral.

Warm Up

Find the following definite integrals.

(a) $\int_{-1}^2 (3x^2 - 2x + 1)dx$

(b) $\int_2^2 (x^2 + 2x - 5)dx$

(c) $\int_{-1}^2 (3x^2 - 6x)dx - \int_{-1}^2 (3x^2 + 4x)dx$

Explanation

(a) $\int_{-1}^{2}(3x^2-2x+1)dx$

$=[x^3-x^2+x]_{-1}^{2}$

$=(2^3-2^2+2)-\{(-1)^3-(-1)^2+(-1)\}$

$=6-(-3)$

$=9$

(b) $\int_{2}^{2}(x^2+2x-5)dx$

$=0$

Upper and lower limits are the same.

(c) $\int_{-1}^{2}(3x^2-6x)dx-\int_{-1}^{2}(3x^2+4x)dx$

$=\int_{-1}^{2}\{(3x^2-6x)-(3x^2+4x)\}dx$

$=\int_{-1}^{2}(-10x)dx$

$=[-5x^2]_{-1}^{2}$

$=(-5\times 2^2)-\{-5\times(-1)^2\}$

$=-15$

If the upper and lower limits of each definite integral are the same, they can be combined into one definite integral.

Try

Find the following definite integrals.

(a) $\int_{1}^{3}(3x^2-4x-1)dx$

(b) $\int_{0}^{2}(3x^2-2x+5)dx$

(c) $\int_{-1}^{3}(x-2)(x+2)dx$

(d) $\int_{3}^{3}(2x^2+x-9)dx$

(e) $\int_{0}^{2}(3x^2+2x-1)dx+\int_{0}^{2}(x-3x^2)dx$

(f) $\int_{-1}^{2}(2x^2+3)dx-\int_{-1}^{2}(3x-1)dx$

Exercise

Find the following definite integrals.

(a) $\int_{-2}^{3} 2x\,dx$

(b) $\int_{1}^{-3} x\,dx$

(c) $\int_{1}^{2} 3x^2\,dx$

(d) $\int_{-2}^{1} (3x^2 + 2x - 1)\,dx$

(e) $\int_{-2}^{1} (6x^2 + 2x - 3)\,dx$

(f) $\int_{-1}^{2} (9x^2 - 4x - 1)\,dx$

(g) $\int_{1}^{1} (4x^2 + x - 3)\,dx$

(h) $\int_{-1}^{3} (3x^2 + 3x - 1)\,dx$

(i) $\int_{-1}^{-2} (2x^2 - 4x + 1)\,dx$

(j) $\int_{-1}^{2} (x^2 - 3x + 2)\,dx$

(k) $\int_{-5}^{-5} (-x^3 - 23x^2 + 19)\,dx$

(l) $\int_{0}^{3} (3x^2 - 4x)\,dx$

(m) $\int_{0}^{1} (x^2 - 2x - 3)\,dx$

(n) $\int_{0}^{2} (2x^2 + 3x + 3)\,dx$

(o) $\int_{-2}^{-1} (x+1)^2\,dx$

(p) $\int_{-1}^{2} (x+3)(x-3)\,dx$

(q) $\int_{-1}^{2} (x-1)(3x+1)\,dx$

(r) $\int_{0}^{0} (3x+2)(2x+3)\,dx$

(s) $\int_{-1}^{3} (x^2 + 2x - 5)\,dx + \int_{-1}^{3} (-x^2 + x + 5)\,dx$

(t) $\int_{-2}^{2} (x^2 + x + 1)\,dx + \int_{-2}^{2} (2x^2 - x - 2)\,dx$

(u) $\int_{-1}^{2} (x^2 + 2x + 2)\,dx + \int_{-1}^{2} (x^2 - x - 1)\,dx$

(v) $\int_{2}^{3} (3x^2 + 6x + 7)\,dx - \int_{2}^{3} (3x^2 + 4x + 3)\,dx$

(w) $\int_{0}^{5} (3x^2 + 2x)\,dx - \int_{0}^{5} (2x^2 + 3x)\,dx$

(x) $\int_{-2}^{1} (x^2 + 5x - 3)\,dx - \int_{-2}^{1} (4x^2 - x - 3)\,dx$

Work with a partner

Collaborate with your group members to answer parts e, f, then discuss the answer.

7-22

MATHEMATICS

Definite Integral (2)

Chapter 7 Differentiation and Integration

Point!

!

Properties of Definite Integrals

(i) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$($k$ is a constant)

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(iii) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

(iv) $\int_\alpha^\beta (x-\alpha)(x-\beta)dx = -\frac{1}{6}(\beta-\alpha)^3$

Warm Up

Find the following definite integrals.

(a) $\int_0^2 (3x^2-4x)dx + \int_2^3 (3x^2-4x)dx$ **(b)** $\int_1^4 (6x^2+4x)dx - \int_3^4 (6x^2+4x)dx$

(c) $\int_{\frac{2}{3}}^1 (3x-2)(x-1)dx$

Learning Outcomes

Calculate the value of the definite integral using the Fundamental Theorem of Calculus.

Swapping the upper and lower limits changes the sign.

If the integrand f(x) is the same, this transformation can be used

$\int_a^c + \int_c^b \Longrightarrow \int_a^b$

This property can be used when the coefficient of x is 1

Explanation

(a) $\displaystyle\int_0^2 (3x^2 - 4x)dx + \int_2^3 (3x^2 - 4x)dx$

$\displaystyle= \int_0^3 (3x^2 - 4x)dx$

$= [x^3 - 2x^2]_0^3$

$= (3^3 - 2 \times 3^2) - 0$

$= 9$

Since the integrand $3x^2 - 4x$ is the same and the range of integration is $0 \to 2$ then $2 \to 3$, Property (iii) can be used.

$\displaystyle\int_0^2 + \int_2^3$

(b) $\displaystyle\int_1^4 (6x^2 + 4x)dx - \int_3^4 (6x^2 + 4x)dx$

$\displaystyle= \int_1^4 (6x^2 + 4x)dx + \int_4^3 (6x^2 + 4x)dx$

$\displaystyle= \int_1^3 (6x^2 + 4x)dx$

$= [2x^3 + 2x^2]_1^3$

$= (2 \times 3^3 + 2 \times 3^2) - (2 \times 1^3 + 2 \times 1^2)$

$= 68$

To use Property (iii), apply Property (ii) to swap the upper and lower limits.

(c) $\displaystyle\int_{\frac{2}{3}}^1 (3x - 2)(x - 1)dx$

$\displaystyle= 3\int_{\frac{3}{2}}^1 \left(x - \frac{2}{3}\right)(x - 1)dx$

$\displaystyle= 3\left[-\frac{1}{6}\left(1 - \frac{2}{3}\right)^3\right]$

$\displaystyle= -\frac{1}{54}$

To use Property (iv), apply Property (i) to make the coefficient of x equal to 1

Try

Find the following definite integrals.

(a) $\displaystyle\int_{-1}^0 (x^2 - 5x + 3)dx + \int_0^1 (x^2 - 5x + 3)dx$

(b) $\displaystyle\int_1^3 (9x^2 - 2x)dx - \int_2^3 (9x^2 - 2x)dx$

(c) $\displaystyle\int_{-\frac{1}{2}}^3 (2x + 1)(x - 3)dx$

Exercise

Find the following definite integrals.

(a) $\int_0^1 (2x^2 - 3x + 5)dx + \int_1^2 (2x^2 - 3x + 5)dx$

(b) $\int_{-1}^0 (x-1)^2 dx + \int_0^1 (x-1)^2 dx$

(c) $\int_0^1 (2x+1)(x+1)dx + \int_{-1}^0 (2x+1)(x+1)dx$

(d) $\int_{-1}^2 (x^2 - x)dx - \int_3^2 (x^2 - x)dx$

(e) $\int_0^{-\frac{1}{3}} (-x^2 - 3x + 2)dx - \int_1^{-\frac{1}{3}} (-x^2 - 3x + 2)dx$

(f) $-\int_2^0 (x+1)(x-1)dx + \int_{-2}^0 (x+1)(x-1)dx$

(g) $\int_3^5 (x-3)(x-5)dx$

(h) $\int_1^2 (x-1)(x-2)dx$

(i) $\int_{-2}^1 (x^2 + x - 2)dx$

(j) $\int_{-1}^{\frac{3}{2}} (x+1)(2x-3)dx$

(k) $\int_{-\frac{1}{3}}^1 (3x+1)(x-1)dx$

(l) $\int_{-1}^{\frac{1}{2}} (2x^2 + x - 1)dx$

Work with a partner

Collaborate with your group members to answer part c, then discuss the answer.

7-23 Applications of Definite Integrals

MATHEMATICS

Chapter 7 Differentiation and Integration

Point!

! If an equation contains a definite integral, let a = (Definite Integral).

Example: If the equation $f(x) = x + \int_0^1 f(t)dt$ is given, let $\mathbf{a} = \int_0^1 \mathbf{f(t)\ dt}$

Learning Outcomes

Calculate the value of the definite integral using the Fundamental Theorem of Calculus.

Warm Up

Find the function $f(x)$ that satisfies the equation $f(x) = x^2 + 2\int_0^1 f(t)dt$

Explanation

Let $a = \int_0^1 f(t)dt$.......**(i)**

Since the equation contains a definite integral, let a = (Definite Integral).

Then, $f(x) = x^2 + 2a$

Replacing x with t gives $f(t) = t^2 + 2a$

Substituting this into **(i)** gives:

$$\int_0^1 f(t)dt = \int_0^1 (t^2 + 2a)dt$$

$$= \left[\frac{1}{3}t^3 + 2at\right]_0^1$$

$$= \left(\frac{1}{3} + 2a\right) - 0$$

$$= \frac{1}{3} + 2a ...... \text{(ii)}$$

From **(i)** and (ii), $a = \frac{1}{3} + 2a$

Solving this gives $a = -\frac{1}{3}$

Therefore, $f(x) = x^2 - \frac{2}{3}$

Try

Find the function $f(x)$ that satisfies the following equations.

(a) $f(x) = x - \int_0^4 f(t)dt$

(b) $f(x) = 3x^2 + 2\int_{-1}^1 f(t)dt$

Exercise

Find the function $f(x)$ that satisfies the following equations.

(a) $f(x) = 3x - \int_0^2 f(t)dt$

(b) $f(x) = 3x + \int_1^3 f(t)dt$

(c) $f(x) = x + 3\int_0^2 f(t)dt$

(d) $f(x) = 3x^2 - 4x + 2\int_0^2 f(t)dt$

(e) $f(x) = 3x^2 - x\int_0^2 f(t)dt + 2$

★ Challenge

Find the function $f(x)$ that satisfies the following equation.

$$f(x) = 3x^2 + \int_0^2 f(t)dt$$

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

7-24

MATHEMATICS

Definite Integrals and Differentiation

Chapter 7 Differentiation and Integration

Point!

If a is a constant, differentiating the function of x, $\int_a^x f(t)dt$, with respect to x gives $\boldsymbol{f(x)}$

Example: Differentiating $\int_3^x (2t^2 + 3t - 5)dt$ with respect to x gives

Replace t with x

$2x^2 + 3x - 5$

Learning Outcomes

Calculate the value of the definite integral using the Fundamental Theorem of Calculus.

Warm Up

Find the function $f(x)$ and the value of constant a that satisfy the equation

$$\int_a^x f(t)dt = x^2 - 4x + 3$$

Explanation

$$\int_a^x f(t)dt = x^2 - 4x + 3 \text{......}\textbf{(i)}$$

Differentiating both sides of **(i)** with respect to x gives:

$$f(x) = 2x - 4$$

Also, substituting a *for* x into **(i)** gives:

$$\int_a^a f(t)dt = a^2 - 4a + 3$$

If the upper and lower limits are the same, the definite integral is 0

$$0 = a^2 - 4a + 3$$

Solving this gives $a = 1,\ 3f(x) = 2x - 4,\ a = 1,\ 3$

Try

Solve the following problems.

(a) Differentiate $\int_{-5}^x (t^2 - t + 3)dt$ with respect to x

(b) Find the function $f(x)$ and the value of constant a that satisfy the equation

$$\int_a^x f(t)dt = x^2 - 3x - 4$$

Exercise

Solve the following problems.

1 Differentiate the following expressions with respect to x

(a) $\int_{3}^{x} (t^2 + 2t + 4)dt$ **(b)** $\int_{-1}^{x} (t^2 - 2t + 3)dt$ **(c)** $\int_{0}^{x} (t^2 - 3t + 7)dt$

2 Find the function f(x) and the value of constant a that satisfy the following equations.

(a) $\int_{a}^{x} f(t)dt = x^2 - 2x - 8$ **(b)** $\int_{a}^{x} f(t)dt = x^2 - 4x + 4$ **(c)** $\int_{1}^{x} f(t)dt = 2x^2 - 3x + a$

Reflect

How does shifting your perspective from studying a static rate of change (derivative) to tracking cumulative accumulation (integral) change the way you analyze real-world motion and growth processes?

★ Challenge

Find the function $f(x)$ and the value of constant a that satisfy the following equations.

(a)

$$\int_{a}^{x} f(t)dt = x^2 - 2x - 8$$

(b)

$$\int_{a}^{x} f(t)dt = x^2 - 4x + 4$$

(c)

$$\int_{1}^{x} f(t)dt = 2x^2 - 3x + a$$

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

7-25

MATHEMATICS

Definite Integrals and Area of Regions (1)

Chapter 7 Differentiation and Integration

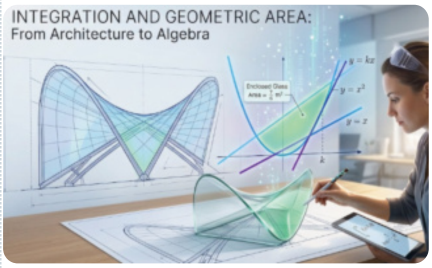
Concept 5 · Applications of Integration to Geometry

In architectural and urban design, structural engineers often calculate the exact surface area of non-linear panels, land parcels, or modern bridges bounded by intersecting curves and tangent lines.

When designing a curved glass facade defined by a parabolic profile and multiple laser-aligned structural beams, determining the precise material footprint requires summing the spaces between curves through definite integration. Guiding Question: How can we calculate the exact geometric area enclosed between multiple intersecting curves, absolute value functions, and tangent lines using definite integrals?

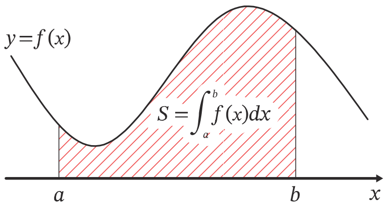
Learning Outcomes

Employ the definite integral to calculate areas bounded under curves, as well as in multiple geometric regions between two curves or straight lines.



Point!

If the graph is above the x–axis

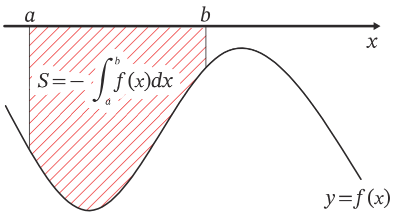


If $f(x) \geq 0$ in the range $a \leq x \leq b$,

The area S of the region bounded by the graph of $y = f(x)$, the x–axis, and the two lines $x = a$ and $x = b$ is given by:

$$S = \int_a^b \mathbf{f(x)\ dx}$$

If the graph is below the x–axis



If $f(x) \leq 0$ in the range $a \leq x \leq b$,

The area S of the region bounded by the graph of $y = f(x)$, the x–axis, and the two lines $x = a$ and $x = b$ is given by:

$$S = -\int_a^b \mathbf{f(x)\ dx}$$

Always draw a graph to find the area. Check whether the bounded region is above or below the x–axis.

Warm Up

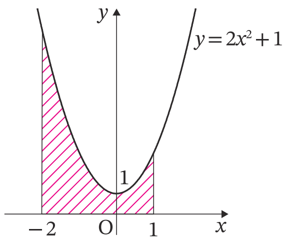
Find the following areas.

1 Area of the region bounded by the parabola $y = 2x^2 + 1$, the x–axis, and the two lines $x = -2$ and $x = 1$

2 Area of the region bounded by the parabola $y = x^2 + x - 6$, the x–axis, and the two lines $x = -2$ and $x = 1$

3 Area of the region bounded by the parabola $y = -x^2 + 4$ and the x–axis.

Explanation



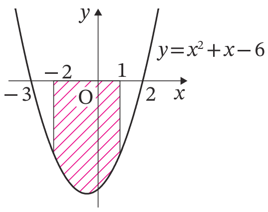
1 Since the graph is as shown on the left, the required area is:

The region is above the x–axis

$$\int_{-2}^{1} (2x^2 + 1)dx$$

$$= \left[\frac{2}{3}x^3 + x\right]_{-2}^{1}$$

$$= 9$$



2 Find the x–coordinates of the x–intercepts of $y = x^2 + x - 6$:

Solving $x^2 + x - 6 = 0$ gives $x = 2,\ -3$

Since the graph is as shown on the left, the required area is:

The region is below the x–axis

$$-\int_{-2}^{1} (x^2 + x - 6)dx$$

$$= \int_{-2}^{1} (-x^2 - x + 6)dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{-2}^{1}$$

$$= \frac{33}{2}$$

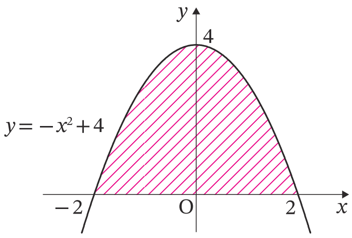
Common Mistakes

$$\int_{-2}^{1} (x^2 + x - 6)dx$$

$$= \left[\frac{1}{3}x^3 + \frac{1}{2}x - 6x\right]_{-2}^{1}$$

$$= -\frac{33}{2}$$

You forgot to include a negative sign even though the graph lies below the x–axis.



3 Find the x—coordinates of the x—intercepts of $y = -x^2 + 4$:

Solving $-x^2 + 4 = 0$ gives $x = \pm 2$

Since the graph is as shown on the left, the required area is:

The region is above the x—axis

$$\int_{-2}^{2} (-x^2 + 4)dx$$

$$= \left[-\frac{1}{3}x^3 + 4x\right]_{-2}^{2}$$

$$= \frac{32}{3}$$

You may use $-\int_{-2}^{2}(x+2)(x-2)dx = -[-\frac{1}{6}\{2-(-2)\}^3]$

Try

Find the areas of the following regions.

1 Area of the region bounded by the parabola $y = x^2 + 3$, the x—axis, and the lines $x = -1$ and $x = 3$

2 Area of the region bounded by the parabola $y = x^2 - 9$, the x—axis, and the lines $x = -2$ and $x = 1$

3 Area of the region bounded by the parabola $y = x^2 - x - 6$ and the x—axis.

4 Area of the region bounded by the parabola $y = -x^2 + 2x$ and the x—axis.

Exercise

Find the areas of the following regions.

1 Area of the region bounded by the parabola $y = x^2 + 2$, the x–axis, and the lines $x = -1$ and $x = 2$

2 Area of the region bounded by the parabola $y = 2x^2 + 2$, the x–axis, and the lines $x = 0$ and $x = 3$

3 Area of the region bounded by the parabola $y = x^2 + 4x + 4$, the x–axis, the y–axis, and the line $x = -1$

4 Area of the region bounded by the parabola $y = 2x^2 - 18$, the x–axis, and the lines $x = -2$ and $x = 2$

5 Area of the region bounded by the parabola $y = x^2 - 6x + 8$, the x–axis, and the lines $x = 2$ and $x = 3$

6 Area of the region bounded by the parabola $y = x^2 - 4$, the x–axis, the y–axis, and the line $x = 1$

7 Area of the region bounded by the parabola $y = x^2 - 16$ and the x–axis.

8 Area of the region bounded by the parabola $y = 3x^2 - 6x$ and the x–axis.

9 Area of the region bounded by the parabola $y = x^2 - 4x - 5$ and the x–axis.

10 Area of the region bounded by the parabola $y = -x^2 + 1$ and the x–axis.

11 Area of the region bounded by the parabola $y = -x^2 + 3x$ and the x–axis.

★ Challenge

A civil engineer is designing a concrete arched retaining wall for a subterranean tunnel entrance. The top curve of the arch follows the parabolic equation $y = -x^2 - 2x + 3$ meters above ground level.

To determine the exact volume of specialty waterproofing sealant required for the front face of the arch, the team needs to calculate the precise surface area enclosed between the arched curve and the ground (x-axis).

Work with a partner

Collaborate with your group members to answer questions 3, 4, then discuss the answer.

7-26

MATHEMATICS

Definite Integrals and Area of Regions (2)

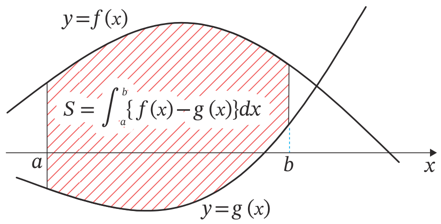
Chapter 7 Differentiation and Integration

Point!

! If $f(x) \geq g(x)$ in the range $a \leq x \leq b$, the area S of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$, and the two lines $x = a$ and $x = b$ is given by:

$$S = \int_a^b \{f(x) - g(x)\}dx$$

Always draw a graph when finding the area. Check which graph is above the other in the bounded region.



Learning Outcomes

Employ the definite integral to calculate areas bounded under curves, as well as in multiple geometric regions between two curves or straight lines.

(Upper graph) – (Lower graph)

Warm Up

Find the areas of the following regions.

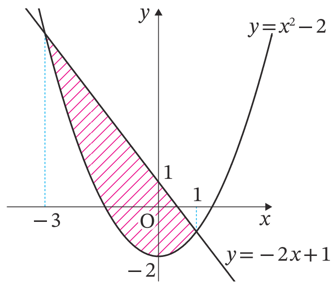
1 Area of the region bounded by the parabola $y = x^2 - 2$ and the line $y = -2x + 1$

2 Area of the region bounded by the two parabolas $y = x^2 + 2x - 4$ and $y = -x^2 + 2x + 4$

3 Area of the region bounded by the parabola $y = -x^2 + 4x(-1 \leq x \leq 2)$ and the three lines $y = x,\ x = -1$, and $x = 2$

Explanation

1



First, find the x—coordinates of the intersections of the parabola and line.

$$\begin{cases} y = x^2 - 2 \\ y = -2x + 1 \end{cases}$$ Solving these gives $x = -3,\ 1$

Only x—coordinates are needed.

Thus, the bounded region is as shown on the left.

From the graph, the required area is:

The graph of $y = -2x + 1$ is the upper boundary.

$$\int_{-3}^{1} \{(-2x+1) - (x^2-2)\}dx$$

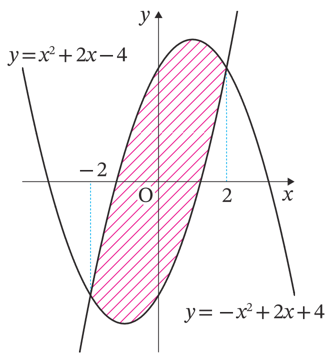
$$= \int_{-3}^{1} (-x^2 - 2x + 3)dx$$

$$= \left[-\frac{1}{3}x^3 - x^2 + 3x\right]_{-3}^{1}$$

$$= \frac{32}{3}$$

You may use $-\int_{-3}^{1}(x+3)(x-1)dx =$ $-[-\frac{1}{6}\{1-(-3)\}^3]$(Refer to 7 – 22)

2



First, find the x—coordinates of the intersections of the two parabolas.

$$\begin{cases} y = x^2 + 2x - 4 \\ y = -x^2 + 2x + 4x \end{cases}$$ Solving these gives =±2

Thus, the bounded region is as shown on the left.

From the graph, the required area is:

The graph of $y = -x^2 + 2x + 4$ is the upper boundary.

$$\int_{-2}^{2} \{(-x^2+2x+4) - (x^2+2x-4)\}dx$$

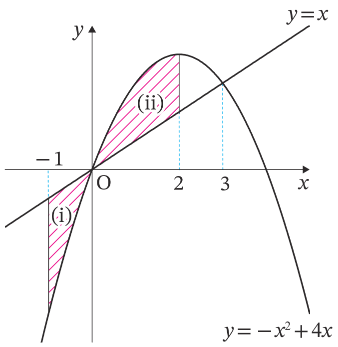
$$= \int_{-2}^{2} (-2x^2 + 8)\mathrm{d}x$$

You may use $-2\int_{-2}^{2}(x+2)(x-2)dx =$ $-2[-\frac{1}{6}\{2-(-2)\}^3]$(Refer to 7 – 22)

$$= \left[-\frac{2}{3}x^3 + 8x\right]_{-2}^{2}$$

$$= \frac{64}{3}$$

3



First, find the x—coordinates of the intersections of the parabola and line.

$$\begin{cases} y = -x^2 + 4x \\ y = x \end{cases}$$ Solving these gives $x = 0,\ 3$

Thus, the bounded region is as shown on the left.

From the graph, the required area is:

$$\underbrace{\int_{-1}^{0} \{x - (-x^2 + 4x)\}dx}_{(i)} + \underbrace{\int_{0}^{2} \{(-x^2 + 4x) - x\}dx}_{(ii)}$$

$$= \int_{-1}^{0} (x^2 - 3x)dx + \int_{0}^{2} (-x^2 + 3x)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_{-1}^{0} + \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_{0}^{2}$$

$$= \frac{31}{6}$$

Calculate the definite integrals separately for the interval where the graph of $y = x$ is above the other, and the interval where $y = -x^2 + 4x$ is above the other.

Try

Find the areas of the following regions.

1 Area of the region bounded by the parabola $y = x^2 - 3x + 2$ and the line $y = 2x - 4$

2 Area of the region bounded by the parabola $y = -x^2 + 3x + 5$ and the line $y = x + 2$

3 Area of the region bounded by the two parabolas $y = x^2 + x - 2$ and $y = -2x^2 + 4x + 4$

4 Area of the region bounded by the parabola $y = x^2 - 3(-1 \leq x \leq 2)$ and the three lines $y = -2x,\ x = -1$, and $x = 2$

Exercise

Find the areas of the following regions.

1 Area of the region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$

2 Area of the region bounded by the parabola $y = x^2 + 2x - 4$ and the line $y = 2x$

3 Area of the region bounded by the parabola $y = x^2 - x - 1$ and the line $y = x + 2$

4 Area of the region bounded by the parabola $y = -2x^2 + 6x + 3$ and the line $y = 2x + 3$

5 Area of the region bounded by the parabola $y = -x^2 + 6$ and the line $y = -x$

6 Area of the region bounded by the parabola $y = -x^2 + 3x - 9$ and the line $y = -2x - 3$

7 Area of the region bounded by the two parabolas $y = x^2 - 3x + 2$ and $y = -x^2 - 3x + 4$

8 Area of the region bounded by the two parabolas $y = 2x^2 - 6x + 4$ and $y = -3x^2 + 9x - 6$

9 Area of the region bounded by the two parabolas $y = x^2 - 3x - 3$ and $y = -x^2 - 2x + 3$

10 Area of the region bounded by the parabola $y = x^2 - 6(-1 \leq x \leq 4)$ and the lines $y = -x,\ x = -1, x = 4$

11 Area of the region bounded by the parabola $y = x^2(-2 \leq x \leq 1)$ and the lines $y = x + 2, x = -2, x = 1$

★ Challenge

A landscape architect is designing a terraced hillside garden. The upper retaining wall of the terrace follows the curved profile given by $y = -x^2 + 8$ meters, spanning horizontally from $x = -2$ to $x = 3$ meters. A diagonal drainage pathway cuts across the slope along the line $y = 2x$.

To order the correct amount of sod for the triangular-like upper section trapped between the curved wall, the drainage line, and the outer boundaries at $x = -2$ and $x = 3$, the installation team needs to calculate the precise surface area of the turf parcel.

Find the total surface area of this garden section.

Work with a partner

Collaborate with your group members to answer questions 3, 4, then discuss the answer.

7-27

MATHEMATICS

Definite Integrals and Area of Regions (3)

Chapter 7 Differentiation and Integration

Learning Outcomes

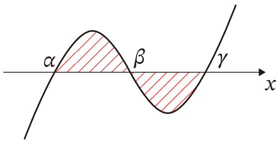
Employ the definite integral to calculate areas bounded under curves, as well as in multiple geometric regions between two curves or straight lines.

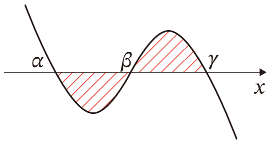
Point!

! For problems finding the area bounded by the graph of a cubic function and the x—axis, also find the x—intercepts and draw the graph.

Let the x—intercepts of the graph of the cubic function $y = ax^3 + bx^2 + cx + d$ be $\alpha,\ \beta,\ \gamma$

If $a > 0$ If $a < 0$

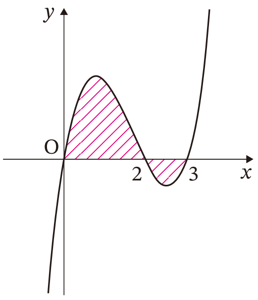




Warm Up

Find the area of the region bounded by the curve $y = x^3 - 5x^2 + 6x$ and the x—axis.

Explanation



Find the x—intercepts of the curve $y = x^3 - 5x^2 + 6x$

Draw the graph of $y = x^3 - 5x^2 + 6x$, which has the shape for $a > 0$

$x^3 - 5x^2 + 6x = 0$

$x\left(x^2 - 5x + 6\right) = 0$

$x(x-2)(x-3) = 0$

$x = 0,\ 2,\ 3$

Since the graph is as shown on the left, the required area is:

$$\underbrace{\int_0^2 (x^3 - 5x^2 + 6x)dx}_{\text{Above the } x\text{—axis}} - \underbrace{\int_2^3 \left(x^3 - 5x^2 + 6x\right) dx}_{\text{Below the } x\text{—axis}}$$

The sign is determined by the positional relationship with the x—axis.

$$= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_0^2 - \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_2^3$$

$$= \frac{8}{3} - 0 - \frac{9}{4} + \frac{8}{3}$$

$$= \frac{37}{12}$$

Try

Find the areas of the following regions.

1. Area of the region bounded by the curve $y = x^3 + x^2 - 6x$ and the x—axis.
2. Area of the region bounded by the curve $y = -x^3 + 5x^2 - 4x$ and the x—axis.

Exercise

Find the areas of the following regions.

1. Area of the region bounded by the curve $y = x^3 - 2x^2 - 3x$ and the x—axis.
2. Area of the region bounded by the curve $y = x^3 - 2x^2 - 8x$ and the x—axis.
3. Area of the region bounded by the curve $y = x^3 - 9x$ and the x—axis.
4. Area of the region bounded by the curve $y = -x^3 + 3x^2 - 2x$ and the x—axis.
5. Area of the region bounded by the curve $y = -x^3 + 7x^2 - 12x$ and the x—axis.

★ Challenge

An environmental engineering team is mapping the cross-sectional profile of an underground pollutant plume moving through a drainage basin. The boundary of the plume's concentration gradient is modeled by the cubic curve $y = -x^3 + 4x$ meters beneath the surface level.

To determine the exact quantity of neutralizing chemical treatment required, the team needs to calculate the total cross-sectional area enclosed between the concentration curve and the baseline ground level (x-axis).

Find the total surface area of the pollutant plume's cross-section.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

7-28

MATHEMATICS

Definite Integrals of Functions with Absolute Values

Chapter 7 Differentiation and Integration

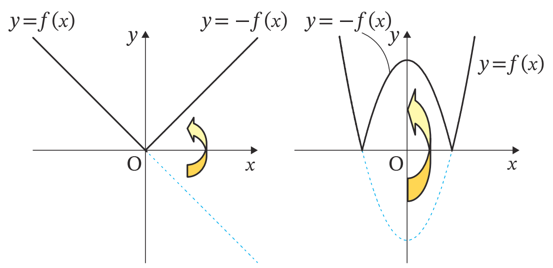
Point!

! $\int_a^b |f(x)|\,dx$ represents the area of the region bounded by the graph of $y = |f(x)|$, the x–axis, and the two lines $x = a$ and $x = b$

How to draw the graph of $y = |f(x)|$:

1. Draw the graph of $y = f(x)$
2. For the graph in 1, fold the portion below the x–axis symmetrically over the x–axis. The equation for the folded portion of the graph becomes

$y = -f(x)$



Learning Outcomes

Apply the rules of integration to functions that contain absolute values after redefining them.

Warm Up

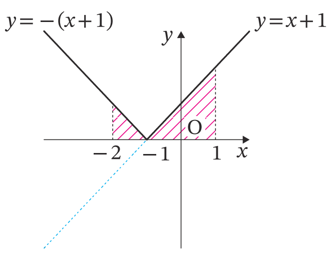
Determine the following definite integrals.

(a) $\int_{-2}^{1} |x+1|\,dx$

(b) $\int_{0}^{3} |x^2-4|\,dx$

Explanation

(a)



The graph of $y = |x+1|$ is as shown on the left.

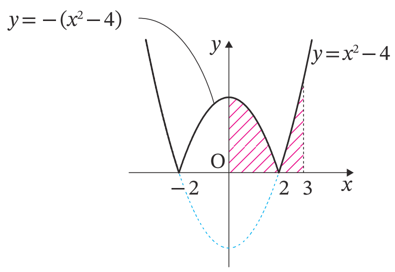
The required definite integral can be found by calculating the sum of the areas of the two parts.

$$\int_{-2}^{-1} \{-(x+1)\}dx + \int_{-1}^{1} (x+1)dx$$

$$= \left[-\frac{1}{2}x^2 - x\right]_{-2}^{-1} + \left[\frac{1}{2}x^2 + x\right]_{-1}^{1}$$

$$= \frac{5}{2}$$

(b)



The graph of $y = |x^2 - 4|$ is shown on the left.

$$\int_0^2 \{-(x^2-4)\}dx + \int_2^3 (x^2-4)dx$$

$$= \left[-\frac{1}{3}x^3 + 4x\right]_0^2 + \left[\frac{1}{3}x^3 - 4x\right]_2^3$$

$$= \frac{23}{3}$$

Try

Find the following definite integrals.

(a) $\int_0^5 |x-3|dx$

(b) $\int_{-1}^3 |x^2-x-2|dx$

Exercise

Find the following definite integrals.

(a) $\int_{-1}^3 |x|\,dx$

(b) $\int_1^3 |x-2|dx$

(c) $\int_{-3}^0 |x+1|dx$

(d) $\int_0^3 |x^2-2x|\,dx$

(e) $\int_{-1}^3 |x^2-1|dx$

★ Challenge

Find the following definite integral.

$$\int_0^2 \left|x^2+3x-4\right|dx$$

Work with a partner

Collaborate with your group members to answer part b, then discuss the answer.

7-29

MATHEMATICS

Tangent Lines and Area

Chapter 7 Differentiation and Integration

Point!

! For problems finding the area bounded by a curve and its tangent line (s), first find the equation(s) of the tangent line (s).

The area bounded by a parabola and two tangent lines is integrated by splitting at the x—coordinate of the intersection of the two tangent lines.

For the area bounded by a curve and one tangent line, find the x—coordinate of the intersection of the curve and the tangent line.

Learning Outcomes

Determine the area bounded by a curve and tangent lines

Warm Up

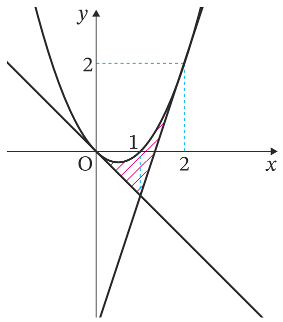
Find the following areas.

1 Area of the region bounded by the parabola $y = x^2 - x$ and the tangent lines at the two points $(0, 0)$ and $(2, 2)$ on it.

2 Area of the region bounded by the curve $y = x^3 + x^2 - 3x + 6$ and the tangent line at the point $(1, 5)$ on it.

Explanation

1



First, find the equations of the tangent lines.

For $y = x^2 - x,\ y' = 2x - 1$

The equation of the tangent line at $(0, 0)$ is $y - 0 = -1(x - 0)$, that is, $y = -x$

The equation of the tangent line at $(2, 2)$ is $y - 2 = 3(x - 2)$, that is, $y = 3x - 4$

Solving $-x = 3x - 4$ gives the x—coordinates of the intersections: $x = 1$

As shown in the figure on the left, the required area is:

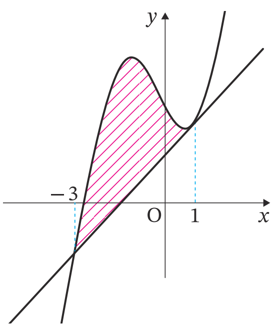
Integrate by splitting at the x—coordinate of the intersection.

$$\int_0^1 \{(x^2 - x) - (-x)\}dx + \int_1^2 \{(x^2 - x) - (3x - 4)\}dx$$

$$= \left[\frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - 2x^2 + 4x\right]_1^2$$

$$= \frac{2}{3}$$

2



First, find the equation of the tangent line.

For $y = x^3 + x^2 - 3x + 6,\ y' = 3x^2 + 2x - 3$

The equation of the tangent line at $(1, 5)$ is $y - 5 = 2(x - 1)$, that is, $y = 2x + 3$

Solving $x^3 + x^2 - 3x + 6 = 2x + 3$ gives the x—coordinates of the intersections:

Since $(1, 5)$ is the point of tangency, factor the expression using $(x - 1)$.

$x = -3,\ 1$

As shown in the figure on the left, the required area is:

$$\int_{-3}^1 \{(x^3 + x^2 - 3x + 6) - (2x + 3)\}dx$$

$$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{5}{2}x^2 + 3x\right]_{-3}^1$$

$$= \left(\frac{1}{4} + \frac{1}{3} - \frac{5}{2} + 3\right) - \left(\frac{81}{4} - 9 - \frac{45}{2} - 9\right)$$

$$= \frac{64}{3}$$

Try

Find the following areas.

1 Area of the region bounded by the parabola $y = x^2 - 4x + 3$ and the tangent lines at the two points $(0, 3)$ and $(4, 3)$.

2 Area of the region bounded by the curve $y = x^3 - 3x^2 + 3x - 1$ and the tangent line at the point $(0, -1)$

Exercise

Find the following areas.

1 Area of the region bounded by the parabola $y = x^2$ and the tangent lines at $(-1, 1)$ and $(2, 4)$

2 Area of the region bounded by the parabola $y = x^2 - 4x$ and the tangent lines at $(1, -3)$ and $(3, -3)$

3 Area of the region bounded by the parabola $y = x^2 + 2x - 3$ and the tangent lines at $(-3, 0)$ and $(1, 0)$

4 Area of the region bounded by the curve $y = x^3 - 3x$ and the tangent line at $(2, 2)$

5 Area of the region bounded by the curve $y = -x^3 + x$ and the tangent line at $(-1, 0)$

6 Area of the region bounded by the curve $y = x^3 + x^2 - 6x - 9$ and the tangent line at $(-2, -1)$

7 Area of the region bounded by the parabola $y = x^2$ and the two tangent lines passing through the point $(1, -3)$

★ Challenge

A civil engineering team is designing a parabolic suspension cable system for a modern pedestrian bridge, modeled by the curve $y = x^2$. Two safety barrier cables are anchored from a utility control station located at coordinates $(3, 8)$ and pulled taut so they form tangent lines touching the parabolic main cable.

To determine the exact volume of protective acoustic coating required for the triangular glass maintenance platform enclosed between the parabolic curve and the two tangent lines, the engineering crew needs to calculate the precise surface area of the region.

Find the total surface area of the maintenance platform.

Work with a partner

Collaborate with your group members to answer Q2, then discuss the answer.

7-30

MATHEMATICS

Bisection of Area

Chapter 7 Differentiation and Integration

Learning Outcomes

Determine the area bounded by a parabola and a line.

Point!

For the area bounded by a parabola and a line, use:

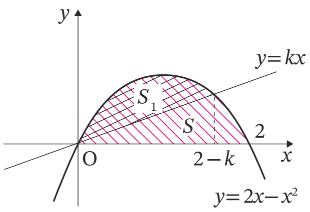
$$\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx = -\frac{1}{6}(\beta-\alpha)^3$$

Warm Up

If the line $y = kx$ bisects the area of the region bounded by the parabola $y = 2x - x^2$ and the x–axis, find the value of the constant k

Explanation

Let the area bounded by the parabola $y = 2x - x^2$ and the x–axis be S,



and the area bounded by the parabola $y = 2x - x^2$ and the line $y = kx$ be S_1.

Then,

$S = 2S_1$......**(i)**

The x–intercepts of the parabola $y = 2x - x^2$ and the x–axis

are found by solving

$2x - x^2 = 0$, giving $x = 0,\ 2$

Thus, $S = \int_0^2 (2x - x^2)dx = -\int_0^2 x(x-2)dx = \frac{1}{6}(2-0)^3 = \frac{4}{3}$......(ii)

Also, the x–coordinates of the intersections of the parabola $y = 2x - x^2$ and the line $y = kx$ are found by solving,

$$\begin{cases} y = 2x - x^2 \\ y = kx \end{cases} \quad \text{giving } x = 0,\ 2-k$$

However, for the line $y = kx$ to bisect the area S,

$0 < 2 - k < 2$, that is, $0 < k < 2$...... (iii)

Thus, $S_1 = \int_0^{2-k} \{(2x - x^2) - kx\}dx$

$$= -\int_0^{2-k} x\left\{x - (2-k)\right\}dx = \frac{1}{6}\left\{(2-k) - 0\right\}^3 = \frac{1}{6}(2-k)^3......(iv)$$

Substituting (ii) and (iv) into (i) gives $\frac{4}{3} = 2 \times \frac{1}{6}(2-k)^3$

$(2-k)^3 = 4$

Since k is a real number, $2 - k = \sqrt[3]{4}$

$k = 2 - \sqrt[3]{4}$

Since this satisfies (iii), $k = 2 - \sqrt[3]{4}$

Try

If the line $y = kx$ bisects the area of the region bounded by the parabola $y = 4x - x^2$ and the x–axis, find the value of the constant k

Exercise

Solve the following problems.

1 If the line $y = kx$ bisects the area of the region bounded by the parabola $y = -x^2 + 8x$ and the x–axis, find the value of the constant k

2 Let the area of the region bounded by the parabola $y = -x^2 + 6x$ and the x–axis be S_1, and the area of the region bounded by this parabola and the line $y = kx$ be S_2. When $S_1 : S_2 = 8:1$, find the value of the constant k.

Reflect

How does understanding the geometric intersection of curves and lines change your perspective on how abstract algebraic equations translate into physical, measurable structures in the real world?

★ Challenge

An architect is designing a parabolic modern art sculpture defined by the curve $y = x^2$. Two laser light beams shoot out from the origin: a baseline beam following $y = x$ and a steeper calibration beam following $y = kx$ (where $k > 1$). To stay within the structural budget, the illuminated glass panel trapped between the parabola and the two beams must have a precise surface area of $\frac{7}{6}$ square meters. Find the exact angle multiplier k required for the calibration beam.

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

8-1

MATHEMATICS

Random Variables and Probability Distributions

Chapter 8 Statistical Inference

Concept 1 · Fundamentals of Random Variables

Consider the process of developing a new life-saving medication. Clinical trials monitor hundreds of patients to observe how many experience a positive recovery, how many report mild side effects, or the precise time it takes for a drug to metabolize in the bloodstream.

Before the trial begins, these exact outcomes are unknown. However, they are not entirely unpredictable. Scientists use mathematical frameworks to assign numerical values to these uncertain outcomes, allowing them to calculate risks, predict efficacy, and ultimately save lives. Guiding Question: How can we mathematically define a function that maps non-numerical outcomes onto a specific set of real numbers while strictly satisfying the laws of probability?

Learning Outcomes

Explain what discrete and continuous random variables are.

Make a probability table for a discrete random variable, read it correctly, and check that all the probabilities add up to 1

Point!

! A value determined by the outcome of a trial, where a probability is assigned to each value, is called a **random variable**.

Example: Number of heads X when tossing 3 coins.

X is determined as one of $0,\ 1,\ 2,\ or\ 3$

The respective probabilities are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$

! The correspondence between the random variable X and the probabilities of taking those values is called a **probability distribution (distribution)**. We say the random variable X **follows** this distribution. To represent a probability distribution, use a **table**.

Example: Probability distribution of X, where X is the number of heads when 3 coins are tossed.

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

! For a random variable X, the probability that $X = a$ is denoted as $P(X = a)$, and the probability that $a \le X \le b$ is denoted as $P\ (a \le X \le b)$

Example: Number of heads X when tossing 3 coins.

$$P\ (x = 0) = \frac{1}{8},\ P\ (0 \le X \le 2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

Warm Up

Two dice are thrown simultaneously. Let X be the sum of the numbers shown.

(a) Find the probability distribution of X. **(b)** Find $P(5 \leq X \leq 8)$

Explanation

(a) For problems with 2 dice, draw a table to find the possible values of X

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Base on the table on the left, the possible values of X are

$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12.$

There is 1 outcome where $X = 2$, so

The probability that $X = 2$ is $\frac{1}{36}$.

The probability that $X = 3$ is $\frac{2}{36}$.

Similarly, find the probability P for each value and create a correspondence table for X and P

X	2	3	4	5	6	7	8	9	10	11	12	Total
P	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

(b) $P\ (5 \leq X \leq 8) = P\ (X = 5) + P\ (X = 6) + P\ (X = 7) + P\ (X = 8)$

$$= \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36}$$

$$= \frac{20}{36}$$

$$= \frac{5}{9}$$

Try

Solve the following problems.

1 Two dice are thrown simultaneously. Let X be the absolute value of the difference between the numbers shown.

(a) Find the probability distribution of X

(b) Find $P(1 \leq X \leq 2)$.

2 From a bag containing5 white balls and 3 red balls, 4 balls are drawn simultaneously. Let X be the number of white balls drawn.

(a) Find the probability distribution of X

(b) Find $P(2 \leq X \leq 3)$

★ Challenge

An automated microfluidic system tests a new compound by taking an initial baseline metabolic reading (0) and executing a sequence of exactly 4 independent chemical micro-injections.

For each micro-injection, the cellular response dictates a numerical shift:

Metabolic Spike (+1 unit) with a probability of 0.5

Metabolic Suppression (−2 units) with a probability of 0.5

Let X represent the net metabolic deviation from baseline recorded after all 4 injections are completed.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Probability of values between 5 and 8

Simplify the fraction for the final answer.

Exercise

Solve the following problems.

1 When 4 coins are thrown simultaneously, let X be the number of heads.

(a) Find the probability distribution of X

(b) Find $P\ (0 \leq X \leq 2)$

2 From a bag containing 4 red balls and 2 blue balls, 3 balls are drawn simultaneously. Let Xbe the number of blue balls drawn.

(a) Find the probability distribution of X

(b) Find $P(1 \leq X \leq 2)$

8-2
MATHEMATICS

Expected Value of Random Variables

Chapter 8 Statistical Inference

Point!

! The sum of the products of the values taken by a random variable and their corresponding probabilities is called **the Expected Value (or Mean)** of the random variable.

The expected value of random variable X is denoted as **E(X)**.

$E(X) = x_1P_1 + x_2P_2 + x_3P_3 + \dots + x_nP_n$

Example: The expected value of X, where X is the number of heads when three coins are tossed.

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$$0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$
$$= \frac{3}{2}$$

! To find the expected value, always draw a **probability distribution table**.

X	x_1	x_2	x_3	…	x_n	Total
P	P_1	P_2	P_3	…	P_n	1

Learning Outcomes

Calculate the expected value of a discrete random variable

Warm Up

Find the expected value $E(X)$ of the following random variable X

(a) Let X be the number of tails if 4 coins are tossed.

(b) Let X be the number of red balls drawn if 3 balls are drawn simultaneously from a bag containing 4 red and 5 white balls.

Explanation

(a) Always draw a probability distribution table.

X	0	1	2	3	4	Total
P	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1

Based on the table on the left, the expected value $E(X)$ is

$$0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}$$
$$= \frac{32}{16}$$
$$= \underline{2}$$

★ Challenge

A regional blood bank manages a highly sensitive emergency supply of 9 plasma units: 5 units of Type O-negative (universal donor plasma) and 4 units of Type AB-negative (rare plasma).

Due to a critical multi-vehicle accident, an emergency response team requests exactly 4 units drawn simultaneously from this specific storage container. Let X represent the number of Type O-negative units included in this emergency draw.

(b) Always draw a probability distribution table.

X	0	1	2	3	Total
P	$\frac{10}{84}$	$\frac{40}{84}$	$\frac{30}{84}$	$\frac{4}{84}$	1

Based on the table on the left, the expected value $E(X)$ is

$$0\times\frac{10}{84}+1\times\frac{40}{84}+2\times\frac{30}{84}+3\times\frac{4}{84}$$
$$=\frac{112}{84}$$
$$=\underline{\frac{4}{3}}$$

Work with a partner

Collaborate with your group members to answer question 3, then discuss the answer.

Try

Find the expected value $E(X)$ of the following random variable X

1 Let X be the number of tails if 3 coins are tossed simultaneously.

2 Let X be the absolute value of the difference between the numbers shown if 2 dice are thrown.

3 Let X be the number of white balls if 3 balls are drawn simultaneously from a bag containing 4 white and 3 red balls.

Exercise

Find the expected value $E(X)$ of the following random variable X

1 There are cards numbered 1 to 7. If one card is drawn face down, let X be the number on that card.

2 Let X be the sum of the numbers shown if 2 dice are thrown.

3 Let X be the number of white balls if 3 balls are drawn simultaneously from a bag containing 3 white and 5 black balls.

4 Among 9 tickets, 2 are winning tickets. Let X be the number of winning tickets if 3 tickets are drawn simultaneously.

5 Let X be the number of heads if 4 coins are tossed simultaneously.

6 Let X be the number of times a 4 is rolled if 1 die is thrown 3 times.

8-3

MATHEMATICS

Properties of Expected Value

Chapter 8 Statistical Inference

Point!

!

If a and b are constants and the probability distribution of random variable X is as shown in the table on the right, the following hold:

X	x_1	x_2	x_3	$\dots$	x_n	Total
P	P_1	P_2	P_3	$\dots$	P_n	1

$E(aX) = aE(X) = a(x_1P_1 + x_2P_2 + x_3P_3 + \dots + x_nP_n)$

$E(aX+b) = aE(X) + b = a(x_1P_1 + x_2P_2 + x_3P_3 + \dots + x_nP_n) + b$

$E\left(X^2\right) = x_1^2P_1 + x_2^2P_2 + x_3^2P_3 + \dots + x_n^2P_n$

Example:

Let X be the number of heads if 3 coins are tossed. The probability distribution of X is shown in the table on the right.

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

Expected value of $3X+2$

$$E(3X+2) = \underline{3\left(0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8}\right) + 2}$$

$$= \frac{13}{2}$$

Expected value of X^2 $\quad E\left(X^2\right) = \underline{0^2\times\frac{1}{8} + 1^2\times\frac{3}{8} + 2^2\times\frac{3}{8} + 3^2\times\frac{1}{8}}$

$= 3$

Learning Outcomes

Apply the algebraic properties of expectation to simplify complex expected value calculations.

Sum of (Square of value of X) × (Probability)

Warm Up

If 4 coins are tossed simultaneously, let X be the number of tails. Find the expected values of the following random variables.

(a) $2X+3$ **(b)** $-4X+1$ **(c)** X^2

Explanation

Always draw a probability distribution table.

X	0	1	2	3	4	Total
P	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1

(a) $E(2X+3) = 2\left(0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}\right) + 3$

$= \underline{7}$

(b) $E(-4X+1) = -4\left(0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}\right) + 1$

$= \underline{-7}$

(c) $E\left(X^2\right) = 0^2 \times \frac{1}{16} + 1^2 \times \frac{4}{16} + 2^2 \times \frac{6}{16} + 3^2 \times \frac{4}{16} + 4^2 \times \frac{1}{16}$

$= \frac{80}{16}$

$= \underline{5}$

Try

If 3 balls are drawn simultaneously from a box containing 4 red balls and 3 white balls, let X be the number of white balls drawn.

Find the expected values of the following random variables.

(a) $3X$ **(b)** $2X + 2$ **(c)** $-3X + 4$ **(d)** X^2

Exercise

Solve the following problems.

1 If 3 coins are tossed simultaneously, let X be the number of heads. Find the expected values of the following random variables.

(a) $-4X$ **(b)** $3X - 1$ **(c)** $-2X + 3$ **(d)** X^2

2 If 2 balls are drawn simultaneously from a bag containing 3 red balls and 2 white balls, let X be the number of white balls drawn. Find the expected values of the following random variables.

(a) $\frac{1}{2}X$ **(b)** $4X + 5$ **(c)** $-\frac{1}{3}X + 2$ **(d)** X^2

★ Challenge

There are five cards numbered 1 to 5. If one card is drawn face down, let X be the number on that card. Find the expected values of the following random variables.

(a) $-2X$ **(b)** $\frac{1}{2}X + 3$

(c) $-3X + 1$ **(d)** X^2

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

8-4
MATHEMATICS

Variance of Random Variables

Chapter 8 Statistical Inference

Point!

! The difference between each value of the random variable X and the expected value m is called the deviation of the random variable X

The expected value of the square of the deviation $(X-m)^2$ is called the **variance** of X, denoted as $V(X)$

$V(X) = E\left((X-m)^2\right)$

! Variance $V(X)$ is a quantity representing how much the values of random variable X are scattered from the expected value $E(X)$

A **smaller** variance indicates that the values of the random variable are clustered closer to the mean.

! V ariance $V(X)$ can also be calculated using the following formula.

$V(X) = \mathrm{E}(\mathrm{X}^2) - \{\mathrm{E}(\mathrm{X})\}^2$

Learning Outcomes

Calculate the variance of a discrete random variable.

(Expected value of X^2) −(Expected value of X)^2

Warm Up

From a box containing 4 red balls and 3 black balls, 2 balls are drawn one by one without replacement. Let X be the number of black balls drawn. Find the expected value $E(X)$ and variance $V(X)$ of the random variable X

Explanation

Always draw a probability distribution table.

X	0	1	2	Total
P	$\frac{12}{42}$	$\frac{24}{42}$	$\frac{6}{42}$	1

Based on the table above,

$$E(X) = 0 \times \frac{12}{42} + 1 \times \frac{24}{42} + 2 \times \frac{6}{42}$$

$$= \frac{36}{42} \quad = \frac{6}{7}$$

$$V(X) = E\left(X^2\right) - \{E(X)\}^2,$$

$$E\left(X^2\right) = 0^2 \times \frac{12}{42} + 1^2 \times \frac{24}{42} + 2^2 \times \frac{6}{42} \quad = \frac{8}{7}$$

Therefore, $V(X) = E\left(X^2\right) - \{E(X)\}^2$

$$= \frac{8}{7} - \left(\frac{6}{7}\right)^2$$

$$= \frac{20}{49}$$

$$E(X) = \frac{6}{7},\ V(X) = \frac{20}{49}$$

Common Mistakes

$$V(X) = E\left(X^2\right) - \{E(X)\}$$
$$= \frac{8}{7} - \frac{6}{7}$$
$$= \frac{2}{7}$$

You forgot to square E[X] in the variance formula.

Try

Find the expected value $E(X)$ and variance $V(X)$ of the following random variable X

1 There are eight cards numbered 1 to 8. If one card is drawn after shuffling, let X be the number on that card.

2 Among 9 tickets, 2 are winning tickets. If3 tickets are drawn simultaneously, let X be the number of winning tickets drawn.

Exercise

Find the expected value $E(X)$ and variance $V(X)$ of the following random variable X

1 There are seven cards numbered 1 to 7. If one card is drawn face down, let X be the number on that card.

2 Among 10 tickets, 3 are winning tickets. If4 tickets are drawn simultaneously, let X be the number of winning tickets drawn.

3 If 3 balls are drawn simultaneously from a box containing 3 white balls and 4 red balls, let X be the number of red balls drawn.

4 There are four cards numbered 1 to 4. If 2 cards are drawn, let X be the sum of the numbers on them.

5 If 2 dice are thrown (one large, one small), let X be the absolute value of the difference between the numbers rolled.

★ Challenge

A biomedical lab keeps a critical test-kit tray containing 9 specialized viral diagnostic assays: 5 assays for Red Flu and 4 assays for Blue Flu.

A clinical technician selects 2 assays, one by one, without replacement to run a patient screening panel. Let X represent the random variable for the number of Blue Flu assays selected for the panel.

Part (a)

Find all possible values of X.

Find the probability distribution of X.

Part (b)

Find the expected number of Blue Flu assays chosen for the panel, $E(X)$.

Find the variance of the chosen Blue Flu assays, $V(X)$, to measure the baseline volatility of this sampling protocol.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

8-5

MATHEMATICS

Standard Deviation of Random Variables

Chapter 8 Statistical Inference

Point!

! The positive square root of the variance $\mathrm{V}(X)$ of a random variable X is called the **standard deviation** of X, denoted as $\sigma(X)$

$\sigma(X) = \sqrt{\mathbf{V(X)}}$

! Like variance, a **smaller** standard deviation indicates that the values of the random variable are clustered closer to the mean.

! Since variance has units that are the square of the original data's units, it cannot be directly compared with the mean.

Standard deviation has the same units as the original data, so it can be compared and calculated with the mean.

Learning Outcomes

Calculate the standard deviation and interpret it as a measure of the spread or risk of a random variable.

Warm Up

If two coins, with the probability of heads being $\frac{4}{7}$ and tails being $\frac{3}{7}$, are tossed simultaneously, let X be the number of tails. Find the standard deviation σ(X)

Explanation

First, find the variance $V(X)$ of X

The probability distribution table of X is as follows.

X	0	1	2	Total
P	$\frac{16}{49}$	$\frac{24}{49}$	$\frac{9}{49}$	1

Based on the probability distribution table, $E(X) = 0 \times \frac{16}{49} + 1 \times \frac{24}{49} + 2 \times \frac{9}{49} = \frac{6}{7}$,

$$E\left(X^2\right) = 0^2 \times \frac{16}{49} + 1^2 \times \frac{24}{49} + 2^2 \times \frac{9}{49} = \frac{60}{49}$$

$V(X) = E\left(X^2\right) - \{E(X)\}^2$

Therefore, $V(X) = E\left(X^2\right) - \{E(X)\}^2 = \frac{60}{49} - \left(\frac{6}{7}\right)^2 = \frac{24}{49}$

Thus, $\sigma(X) = \sqrt{\frac{24}{49}} = \frac{2\sqrt{6}}{7}$

Try

Solve the following problems.

1 If the probability distribution of random variable X is given by the following table, find the standard deviation $\sigma(X)$

(a)

X	1	2	3	Total
P	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	1

(b)

X	0	1	2	3	Total
P	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$	1

2 If 2 balls are drawn simultaneously from a box containing 4 red balls and 3 white balls, let X be the number of white balls drawn. Find the standard deviation σ(X)

Exercise

Solve the following problems.

1 If the probability distribution of random variable X is given by the following table, find the standard deviation $\sigma(X)$

(a)

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

(b)

X	0	1	2	3	4	Total
P	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1

★ Challenge

An agricultural engineer utilizes a smart irrigation system equipped with 8 moisture sensors distributed across a farming field. Due to a sudden malfunction in the wireless communication network, some sensors have stopped transmitting data correctly. The system logs reveal that exactly 2 sensors are operating at high efficiency, while the remaining 6 sensors are transmitting faulty readings.

The system automatically pulls a random sample of 4 sensors simultaneously to recalibrate them and test their connectivity quality. Let X represent the random variable for the number of functional (highly efficient) sensors selected in this emergency sample.

Part (a)

Find all possible values of the random variable X.

Find the complete probability distribution of X.

Part (b)

Find the standard deviation $\sigma(X)$ of the number of functional sensors selected, in order to evaluate the precision and stability of the automatic recalibration system.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

8-6
MATHEMATICS

Transformation of Random Variables (1)

Chapter 8 Statistical Inference

Point!

! For a random variable X, the linear expression $aX+b$ is also a random variable, and the following equations hold.

Expected value of $aX+b$ $\quad E(aX+b)=$ $\mathrm{aE}(\mathbf{X})+\mathrm{b}$

Variance of $aX+b$ $\quad V(aX+b)=$ $\mathbf{a}^2\mathrm{V}(\mathbf{X})$

Standard deviation of $aX+b$ $\quad \sigma(aX+b)=$ $|\mathbf{a}|\sigma(\mathbf{X})$

Learning Outcomes

Apply transformations involving multiple constants or non-linear adjustments to random variables and correctly compute the resulting mean and variance.

Warm Up

Among 10 tickets, 3 are winning tickets. If 4 tickets are drawn simultaneously, let X be the number of winning tickets drawn. Find the variance $\mathrm{V}(Y)$ and standard deviation $\sigma(Y)$ of the following random variable Y

(a) $Y=2X+1$ **(b)** $Y=-3X-2$

Explanation

First, draw the probability distribution table of X and find $V(X)$ and $\sigma(X)$

X	0	1	2	3	Total
P	$\frac{35}{210}$	$\frac{105}{210}$	$\frac{63}{210}$	$\frac{7}{210}$	1

Based on the probability distribution table, $E(X)=0\times\frac{35}{210}+1\times\frac{105}{210}+2\times\frac{63}{210}+3\times\frac{7}{210}$

$$=\frac{6}{5}$$

$$E(X^2)=0^2\times\frac{35}{210}+1^2\times\frac{105}{210}+2^2\times\frac{63}{210}+3^2\times\frac{7}{210}$$

$$=2$$

$$V(X)=E(X^2)-\{E(X)\}^2=2-\left(\frac{6}{5}\right)^2$$

$$=\frac{14}{25}$$

$$\sigma(X)=\sqrt{V(X)}=\sqrt{\frac{14}{25}}$$

$$=\frac{\sqrt{14}}{5}$$

Work with a partner

Collaborate with your group members to answer try section, then discuss the answer.

(a) For $Y = 2X + 1$,

$V(2X+1) = 2^2 \times V(X)$ $\qquad \sigma(2X+1) = |2| \times \sigma(X)$

$= 4 \times \dfrac{14}{25}$ $= 2 \times \dfrac{\sqrt{14}}{5}$

$= \underline{\dfrac{56}{25}}$ $= \underline{\dfrac{2\sqrt{14}}{5}}$

(b) For $Y = -3X - 2$,

$V(-3X-2) = (-3)^2 \times V(X)$ $\qquad \sigma(-3X-2) = |-3| \times \sigma(X)$

$= 9 \times \dfrac{14}{25}$ $= 3 \times \dfrac{\sqrt{14}}{5}$

$= \underline{\dfrac{126}{25}}$ $= \underline{\dfrac{3\sqrt{14}}{5}}$

Try

If 2 balls are drawn simultaneously from a bag containing 3 red balls and 4 white balls, let X be the number of red balls drawn. Find the variance $V(Y)$ and standard deviation $\sigma(Y)$ of the following random variable Y

(a) $Y = X + 3$ **(b)** $Y = 2X - 5$ **(c)** $Y = -3X + 4$

Exercise

Solve the following problems.

1 Let X be the number rolled if a die is thrown once. Find the variance $V(Y)$ and standard deviation $\sigma(Y)$ of the following random variable Y

(a) $Y = X + 2$ **(b)** $Y = 3X + 1$ **(c)** $Y = -2X + 4$

★ Challenge

A logistics tech startup launches exactly 3 independent delivery drones simultaneously to test a new high-speed route. For each drone, there is a 50% (0.5) chance it finishes ahead of schedule and a 50% (0.5) chance it finishes on time.

Let X represent the random variable for the number of drones that finish ahead of schedule.

The operations team evaluates three different financial risk models (Y) to determine how performance fluctuations affect their daily delivery payouts. Find the variance

V(Y) and standard deviation

$\sigma(Y)$ of the following random variable Y.

(a) $Y = X - 3$

(b) $Y = 2X - 1$

(c) $Y = -3X + 1$

8-7 MATHEMATICS

Transformation of Random Variables (2)

Chapter 8 Statistical Inference

Point!

! For a random variable X, the linear expression $aX+b$ is also a random variable, and the following equations hold.

Expected value of $aX+b$ $\quad E(aX+b)=$ $\underline{aE(X)+b}$

Variance of $aX+b$ $\quad V(aX+b)=$ $\underline{a^2V(X)}$

Standard deviation of $aX+b$ $\quad \sigma(aX+b)=$ $\underline{|a|\sigma(X)}$

Learning Outcomes

Apply transformations involving multiple constants or non-linear adjustments to random variables and correctly compute the resulting mean and variance.

Warm Up

Solve the following problems.

(a) There is a game where you toss two coins and receive 50EGP for each head. If you play this game once by paying 30 EGP entry fee, find the expected value and the standard deviation of the net prize (the prize amount after deducting the entry fee).

(b) Let X be the number of tails obtained if3 coins are tossed simultaneously. Let $Y=aX+b$. Find the values of a and b such that the expected value of Y is $7\sqrt{3}$ and the standard deviation is 3. Assume $a>0$

Explanation

(a) Let X be the number of heads. The probability distribution is shown in the table below.

X	0	1	2	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

Based on the table, $E(X)=1, E\left(X^2\right)=\frac{3}{2}$, so,

$$V(X)=E\left(X^2\right)-\{E(X)\}^2$$

$$=\frac{3}{2}-1^2=\frac{1}{2}$$

$$\sigma(X)=\sqrt{V(X)}=\frac{\sqrt{2}}{2}$$

Next, let Y be the prize money after deducting the entry fee,

Y is expressed as $Y=50X-30$

It is expressed in the form $Y=aX+b$

Therefore, the expected value of Y is: $E(Y)=E(50X-30)=50\times 1-30=20$

The standard deviation of Y is: $\sigma(Y)=\sigma(50X-30)=|50|\times\frac{\sqrt{2}}{2}=25\sqrt{2}$

$\underline{\text{Expected value: 20, Standard deviation: } 25\sqrt{2}}$

(b) The probability distribution of X is shown in the table below.

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

Based on the table, $E(X) = \frac{3}{2}$, $E\left(X^2\right) = 3$,

$$V(X) = E\left(X^2\right) - \{E(X)\}^2$$

$$= 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$\sigma(X) = \sqrt{V(X)} = \frac{\sqrt{3}}{2}$

Since $E(aX + b) = 7\sqrt{3}$, $aE(X) + b = 7\sqrt{3}$

$\frac{3}{2}a + b = 7\sqrt{3}$ ......**(i)**

Also, since, $\sigma(aX + b) = 3$, $|a|\sigma(X) = 3$,

σ(aX + b) = |a|σ(X)

$\frac{\sqrt{3}}{2}\,|\,a\,| = 3$, Since $a > 0$, $\frac{\sqrt{3}}{2}a = 3$ ...... (ii)

Solving (i) and (ii), $\underline{a = 2\sqrt{3},\ b = 4\sqrt{3}}$

Try

Solve the following problems.

1 There is a game where you draw 2 balls simultaneously from a box containing 4 red balls and 3 black balls, receiving 20 EGP for each black ball drawn. If you pay 10 EGP to play this game once, find the expected value and standard deviation of the net winnings.

2 Let X be the number obtained if a die is thrown once. Let $Y = aX + b$. Find the values of a and b such that the expected value of Y is 0 and the standard deviation is10 Assume $a > 0$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Exercise

Solve the following problems.

1 Point P starts from the origin O and moves on a number line. P moves $+2$if a coin lands heads and -1 if it lands tails. Find the expected value and variance of the coordinate of P after tossing the coin 3 times.

2 There is a game where you draw 3 balls simultaneously from a bag containing 5 white balls and 4 red balls, receiving 30 EGP for each red ball drawn . If you pay 20 EGP to play this game once, find the expected value and standard deviation of the net winnings.

3 There is a game where you toss two 100 pt coins and receive an amount equal to the value of the coins that land heads up. If you pay 100 pt to play this game once, find the expected value and standard deviation of the net winnings.

4 Let X be the number obtained if a die is thrown once. Let $Y = aX + b$. Find the values of a and b such that the expected value of Y is $3\sqrt{21}$ and the standard deviation is $\sqrt{5}$ Assume $a > 0$

★ Challenge

A green-energy startup operates a microgrid powered by two independent wind turbines. Because wind conditions fluctuate, each turbine has a 50% (0.5) chance of operating at peak performance and a 50% (0.5) chance of operating at idle baseline.

Let X represent the random variable for the number of turbines operating at peak performance during a given hour.

To stabilize the local grid, the energy is directed through an automated smart inverter that scales the voltage output. The final power distribution to the grid, measured in megawatts (MW), is modeled by the linear transformation:

$$Y = aX + b$$

The engineering team needs to calibrate the system variables (a and b) to meet specific safety and efficiency standards.

The Assessment

Part (a): Find the expected value $E(X)$ and the standard deviation $\sigma(X)$ of the baseline peak turbine performance.

Part (b): Find the exact calibration values for the tuning coefficients a and b such that the final grid power output Y maintains an expected value $E(Y) = 12\sqrt{2}$ MW and a standard deviation of 8 MW. (Assume the scale factor $a > 0$.)

Reflect

How does changing our perspective from tracking individual, unpredictable outcomes to analyzing the collective probability distribution of a random variable change our ability to make decisions in science and society?

8-8

MATHEMATICS

Expected Value of Multiple Random Variables (1) (Sum)

Chapter 8 Statistical Inference

Concept 2 · Expected Value and Variance of Multiple Variables

Think about managing a busy high school cafeteria or a local fast-food restaurant. Every day, the kitchen manager has to predict how much food to prepare. Let's look at two popular lunch items: chicken wraps (let's call this variable X) and fruit smoothies (variable Y). The number of wraps sold changes every day depending on the weather or what day of the week it is, but the manager knows the historic daily average.

The number of smoothies sold also fluctuates on its own. If the manager wants to plan the total daily lunch revenue, they have to look at both items together (X + Y). If a massive heatwave hits, students might buy fewer warm wraps but way more cold smoothies, meaning the sales of one item directly affect the other. How do these individual daily fluctuations combine to affect the total overall daily revenue? Understanding how multiple random variables behave together allows businesses, hospitals, and scientists to plan for the future, budget accurately, and manage unpredictable changes.

Inquiry Question: When we add two random variables, how do the rules of expectation work, and under what condition do their variances simply add up?

Learning Outcomes

Calculate the expected value of the sum of two or three random variables

Point!

For random variables X and Y, their sum $X+Y$ is also a random variable, and the following holds.

$$E(X+Y) = \mathbf{E(X)} + \mathbf{E(Y)}$$

For 3 or more random variables, the following similarly holds.

$$E(X+Y+Z) = \mathbf{E(X)} + \mathbf{E(Y)} + \mathbf{E(Z)}$$

It is easier to add the individual expected values than to calculate the expected value of the sum.

Warm Up

Solve the following problems.

(a) Given the probability distributions of random variables X and Y in the following tables, find the expected value of $X+Y$

X	5	10	20	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Y	3	6	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

(b) If a 1-pound coin, a 50-piastre coin, and a 25-piastre coin are tossed once each, let X, Y, and Z be the number of heads for each coin respectively. Find $E(X+Y+Z)$

Explanation

(a) Based on the probability distribution tables,

$$E(X) = 5 \times \frac{1}{3} + 10 \times \frac{1}{3} + 20 \times \frac{1}{3} = \frac{35}{3}$$

$$E(Y) = 3 \times \frac{1}{2} + 6 \times \frac{1}{2}$$

$$= \frac{9}{2}$$

Therefore, $E(X+Y) = E(X) + E(Y)$

$$= \frac{35}{3} + \frac{9}{2}$$

$$= \underline{\frac{97}{6}}$$

(b) Since $E(X+Y+Z) = E(X) + E(Y) + E(Z)$, find the expected values of X, Y, and Z respectively.

The probability distributions are all the same, as shown in the table below.

X	0	1	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

Y	0	1	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

Z	0	1	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

Based on the table above, $E(X) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

Similarly, $E(Y) = \frac{1}{2}$, $E(Z) = \frac{1}{2}$

Therefore, $E(X+Y+Z) = E(X) + E(Y) + E(Z)$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \underline{\frac{3}{2}}$$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Try

Solve the following problems.

1 Given the probability distributions of random variables X and Y in the following tables, find the expected value of $X + Y$

X	1	2	3	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Y	4	5	6	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

2 If three dice (large, medium, and small) are thrown, let X, Y, and Z be the numbers rolled on each die respectively. Find $E(X+Y+Z)$

Exercise

Solve the following problems.

1 Given the probability distributions of random variables X, Y, and Z in the following tables, find the expected value of $X + Y + Z$

X	1	4	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

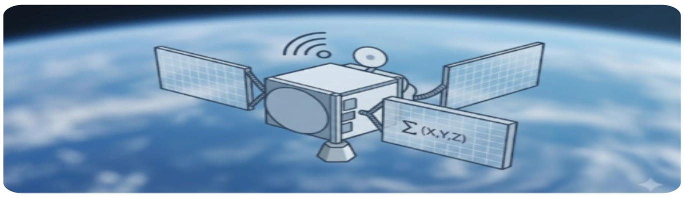
Y	2	5	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

Z	3	6	Total
P	$\frac{1}{2}$	$\frac{1}{2}$	1

2 Bag A contains 4 white balls and 3 red balls, and Bag B contains 5 white balls and 2 red balls. Let X be the number of red balls drawn if 2 balls are drawn from Bag A, and Y be the number of red balls drawn if 2 balls are drawn from Bag B. Find $E(X + Y)$

3 If a 1-pound coin, a 50-piastre coin, and a 25-piastre coin are tossed twice each, let X, Y, and Z be the number of heads for each respectively. Find $E(X + Y + Z)$

★ Challenge



A micro-satellite relies on three independent sub-systems: X (Thrusters), Y (Solar Panels), and Z (Comms). During an orbital test, each sub-system is sent exactly two activation signals ($n = 2$). Due to static interference, each signal has a 50% success rate ($p = 0.5$). Let X, Y, and Z be the number of successful activations for each sub-system. Find the total expected number of successful system activations, E($X + Y + Z$).

8-9

MATHEMATICS

Expected Value of Multiple Random Variables (2) (aX + bY)

Chapter 8 Statistical Inference

Point!

! If a and b are non—zero constants, the following holds for random variables X and Y

$E(aX + bY) = aE(X) + bE(Y)$

Learning Outcomes

Use the linearity of expectation to compute the expected value of a linear combination of random variables

Warm Up

In a certain game, two 500-point coins and three 100-point coins are tossed simultaneously. Each coin has an equal probability of landing on "heads" or "tails." Let Z be the total points of the coins that land on heads. Find the expected value $E(Z)$

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Explanation

Let X be the number of 500-point coins that land on heads, and Y be the number of 100-point coins that land on heads.

To find the expected values of X and Y, first, let's summarize their respective probability distributions in tables.

X	0	1	2	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

Y	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

Based on the probability distribution tables, $E(X) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4}$

$= \frac{4}{4}$

$= 1$

$E(Y) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{12}{8}$

$= \frac{3}{2}$

Here, Z can be expressed as $Z = 500X + 100Y$,

Express in the form $Z = aX + bY$

$E(Z) = E(500X + 100Y)$

$= 500 \times E(X) + 100 \times E(Y)$

$= 500 \times 1 + 100 \times \frac{3}{2}$

$= 650$

Try

Solve the following problems.

1 In a certain game, two 500-point coins and two 100-point coins are tossed simultaneously. Each coin has an equal probability of landing on "heads" or "tails." Let Z be the total points of the coins that land on heads. $E(Z)$

2 If a die is thrown twice, the score is 3 times the number rolled on the first roll and 5 times the number rolled on the second roll. Let Z be the total score of the two rolls. Find the expected value of Z

Exercise

Solve the following problems.

1 If three 100-pt coins and one 50-pt coin are tossed simultaneously, let Z be the sum of the value of coins landing on heads. Find the expected value of Z

2 If a die is thrown twice, the score is 2 times the number rolled on the first roll and 10 times the number rolled on the second roll. Let Z be the total score. Find $E(Z)$

3 Box A contains 5 white balls and 4 red balls, and Box B contains 4 white balls and 2 red balls. Let Z be the sum of the number of red balls drawn if 2 balls are drawn simultaneously from Box A and 2 balls are drawn simultaneously from Box B. Find $E(Z)$

★ Challenge

Imagine your company is hosting a charity gala with two separate raffles:

Raffle A (The Tech Basket): has 10 total tickets, and 2 of them are winners.

Raffle B (The Wellness Basket): has 6 total tickets, and only 1 is a winner.

You decide to support the cause and buy 2 tickets from Raffle A and 2 tickets from Raffle B. Let Z be the total number of winning tickets you take home. What is the expected number of prizes you will win?

8-10
MATHEMATICS

Expected Value of Multiple Random Variables (3) (Product)

Chapter 8 Statistical Inference

Point!

For all values a and b taken by two random variables X and Y, when $P(X = a,\ Y = b) = P(X = a) \times P(Y = b)$ always holds, regardless of the values of a and b, the random variables X and Y are said to be **mutually independent**.

Example: If 3 coins are tossed and a die is rolled, let X be the number of heads and let Y be the number rolled. X and Y are mutually independent.

If random variables X and Y are **mutually independent**,

$E(XY) = $ **E(X) E(Y)**

Learning Outcomes

Apply the property to calculate the expected value of the product of two independent random variables explain why independence is necessary for this rule to hold.

If the occurrence of $X = a$ and the occurrence of $Y = b$ do not affect each other, (Probability of $X = a$ and $Y = b$) = (Probability of $X = a$) × (Probability of $Y = b$).

The formula for the expected value of a product requires the condition "mutually independent".

Warm Up

Solve the following problems.

1 Given that two random variables X and Y are mutually independent and their probability distributions are given in the tables on the right, find $E(XY)$

X	1	3	5	Total
P	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	1

Y	2	4	6	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

2 Ahmed tosses 2 coins simultaneously once, and Mohamed throws a die once. Let X be the number of heads and Y be the number rolled. Find the expected value of the product XY

Explanation

1 Since random variables X and Y are mutually independent, $E(XY) = E(X)E(Y)$

First, find $E(X)$ and $E(Y)$

Based on the probability distribution tables, $E(X) = 1 \times \frac{2}{5} + 3 \times \frac{2}{5} + 5 \times \frac{1}{5} = \frac{13}{5}$

$$E(Y) = 2 \times \frac{1}{4} + 4 \times \frac{2}{4} + 6 \times \frac{1}{4} = 4$$

Therefore, $E(XY) = E(X)E(Y) = \frac{13}{5} \times 4 = \underline{\frac{52}{5}}$

2 Since the trial of tossing coins and the trial of throwing a die are mutually independent, random variables X and Y are also mutually independent.

Therefore, $E(XY) = E(X)E(Y)$

The probability distributions for each trial are shown in the tables below.

X	0	1	2	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

Y	1	2	3	4	5	6	Total
P	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

Based on the tables, $E(X) = 1, E(Y) = \frac{7}{2}$

Thus, $E(XY) = E(X)E(Y) = 1 \times \frac{7}{2} = \underline{\frac{7}{2}}$

Try

Solve the following problems.

X	1	2	Total
P	$\frac{3}{4}$	$\frac{1}{4}$	1

Y	3	4	Total
P	$\frac{2}{3}$	$\frac{1}{3}$	1

1 Given that two random variables X and Y are mutually independent and their probability distributions are given in the table on the right, find $E(XY)$

2 If a die and a coin are thrown and tossed simultaneously once each, let X be the number rolled and Y be the number of heads. Find the expected value of the product XY

Exercise

Solve the following problems.

1 Given that two random variables$X$ and Y are mutually independent and their probability distributions are given in the following tables, find $E(XY)$

(a)

X	1	3	Total
P	$\frac{1}{3}$	$\frac{2}{3}$	1

Y	2	4	Total
P	$\frac{3}{4}$	$\frac{1}{4}$	1

(b)

X	1	2	3	Total
P	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

Y	1	2	Total
P	$\frac{2}{5}$	$\frac{3}{5}$	1

(c)

X	0	2	4	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Y	1	3	5	Total
P	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	1

2 Ahmed tosses 3 coins and Mohamed tosses 2 coins simultaneously. Let X and Y be the number of heads for each respectively. Find $E(XY)$

★ Challenge

There are four cards numbered 1 to 4. If one card is drawn, put back, and another card is drawn, let X be the number on the first card and Y be the number on the second card. Find the expected value of the product XY.

8-11
MATHEMATICS

Variance of Sum of Random Variables

Chapter 8 Statistical Inference

Point!

If random variables X and Y are **mutually independent**,

$V(X+Y) = \mathbf{V(X)} + \mathbf{V(Y)}$

$\sigma(X+Y) = \sqrt{\mathrm{V(X+Y)}}$

Learning Outcomes

Calculate the variance of the sum for independent variables, and extend this to linear combinations.

The formula for variance requires the condition "mutually independent".

Warm Up

If there are 3 cards numbered 1 to 3 and one card is drawn, let X be the number on it. Next, if 3 coins are tossed simultaneously, let Y be the number of heads. Find $V(X+Y)$ and $\sigma(X+Y)$

Explanation

Since the trial of drawing a card and the trial of tossing coins are mutually independent, random variables X and Y are also mutually independent.

The probability distributions for each trial are shown in the tables below.

X	1	2	3	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Y	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

First, find the variance for each.

$E(X) = 2,\ E(X^2) = \frac{14}{3}$, so $V(X) = E(X^2) - \{E(X)\}^2 = \frac{14}{3} - 2^2 = \frac{2}{3}$

$E(Y) = \frac{3}{2},\ E(Y^2) = 3$, so $V(Y) = E(Y^2) - \{E(Y)\}^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

Therefore, $V(X+Y) = V(X) + V(Y) = \frac{2}{3} + \frac{3}{4}$

$= \frac{17}{12}$

$\sigma(X+Y) = \sqrt{V(X+Y)} = \sqrt{\frac{17}{12}}$

$= \frac{\sqrt{51}}{6}$

Try

Solve the following problems.

X	1	2	3	Total
P	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	1

Y	1	3	5	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

1 Given that two random variables X and Y are mutually independent and their probability distributions are given in the table on the right, find $V(X+Y)$ and $\sigma(X+Y)$

2 If two dice (large and small) are thrown, let X be the number rolled on the large die and Y be the number rolled on the small die. Find the variance and standard deviation of $X+Y$

Exercise

Solve the following problems.

1 Given that two random variables X and Y are mutually independent and their probability distributions are given in the following tables, find $V(X+Y)$ and $\sigma(X+Y)$

(a)

X	1	2	Total
P	$\frac{3}{5}$	$\frac{2}{5}$	1

Y	3	4	Total
P	$\frac{3}{4}$	$\frac{1}{4}$	1

(b)

X	1	3	5	Total
P	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	1

Y	2	4	6	Total
P	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	1

(c)

X	0	3	6	Total
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

Y	1	4	Total
P	$\frac{2}{5}$	$\frac{3}{5}$	1

2 If one coin is tossed and one die is thrown, let X be the number of heads and Y be the number rolled. Find the variance and standard deviation of $X+Y$

3 Ahmed tosses 2 coins and Mohamed tosses 3 coins simultaneously. Let X and Y be the number of heads for each respectively. Find $V(X+Y)$ and $\sigma(X+Y)$

★ Challenge

Imagine you are a retail investor managing a small, algorithmic trading portfolio. To diversify, your automated system splits your capital into two entirely independent market metrics each morning:

The Market Volatility Index (X): The system selects one of 4 risk levels (scaled uniformly from 1 to 4) based on early pre-market futures.

The Crypto Sector Momentum (Y): A high-frequency crypto trading bot yields a daily risk factor represented by an automated scale from 1 to 6 (modeled as a standard 6-sided die roll).

Your total portfolio exposure for the day is defined as $Z = X + Y$. To allocate your cash reserves properly and ensure you don't breach your risk tolerance, your risk management software needs to calculate the total variance and standard deviation of this combined exposure.

Work with a partner

Collaborate with your group members to answer question 2, then discuss the answer.

8-12

MATHEMATICS

Three Random Variables

Chapter 8 Statistical Inference

Point!

! If random variables $X,\ Y$, and Z are **mutually independent**,

$E(XYZ) = \mathbf{E(X)\,E(Y)\,E(Z)}$

$V(X+Y+Z) = \mathrm{V}(\mathbf{X}) + \mathrm{V}(\mathbf{Y}) + \mathrm{V}(\mathbf{Z})$

Learning Outcomes

Extend the formulas for expectation and variance to linear combinations involving three random variables, under independence.

Warm Up

If three dice, each having two faces marked with 2, two faces marked with 4, and two faces marked with 6, are thrown simultaneously, find the following values.

(a) Expected value of the product of the numbers rolled.

(b) Variance of the sum of the numbers rolled.

Explanation

Let $X,\ Y$, and Z be the numbers rolled on each die.

The probability distributions are all the same, as shown in the table below.

X	2	4	6	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Y	2	4	6	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Z	2	4	6	Total
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

(a) $E(X) = 2\times\frac{1}{3} + 4\times\frac{1}{3} + 6\times\frac{1}{3} = 4$

Similarly for $E(Y)$ and $E(Z)$,

Expected value of the product $E(XYZ) = E(X)E(Y)E(Z)$

$= 4\times4\times4$

$= \underline{64}$

Work with a partner

Collaborate with your group members to answer question b, then discuss the answer.

(b) $E\left(X^2\right) = 2^2 \times \frac{1}{3} + 4^2 \times \frac{1}{3} + 6^2 \times \frac{1}{3} = \frac{56}{3}$

$$V(X) = E\left(X^2\right) - \{E(X)\}^2$$
$$= \frac{56}{3} - 4^2$$
$$= \frac{8}{3}$$

Similarly for $V(Y)$ and $V(Z)$, so

Variance of the sum $V(X + Y + Z) = V(X) + V(Y) + V(Z)$

$$= \frac{8}{3} + \frac{8}{3} + \frac{8}{3}$$
$$= \underline{8}$$

Try

If three special dice are thrown simultaneously, where each die has the numbers 1, 3, and 5 marked on two of its faces, find the following values. Find the following values.

(a) Expected value of the product of the numbers rolled

(b) Variance of the sum of the numbers rolled

Exercise

Solve the following problems.

1 If three dice (large, medium, small) are thrown simultaneously, find the following values.

(a) Expected value of the product of the numbers rolled

(b) Variance of the sum of the numbers rolled

2 If Person A tosses 4 coins, Person B tosses 3 coins, and Person C tosses 2 coins simultaneously, let X, Y, and Z be the number of heads for each respectively. Find the following values.

(a) $E(XYZ)$　　**(b)** $V(X + Y + Z)$

Reflect

Why can it be risky to assume two random variables are independent when calculating total variance, and how can we check if they really are independent?

★ Challenge

Three friends play a dice game where they try to roll a "6" to score points.

X gets 5 rolls.

Y gets 4 rolls.

Z gets 3 rolls.

They roll independently. Find:

(a) $\mathrm{E}(XYZ)$.

(b) $\mathrm{V}(X + Y + Z)$.

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