Mathematics II

Egyptian Baccalaureate

2nd Year

Part One

11

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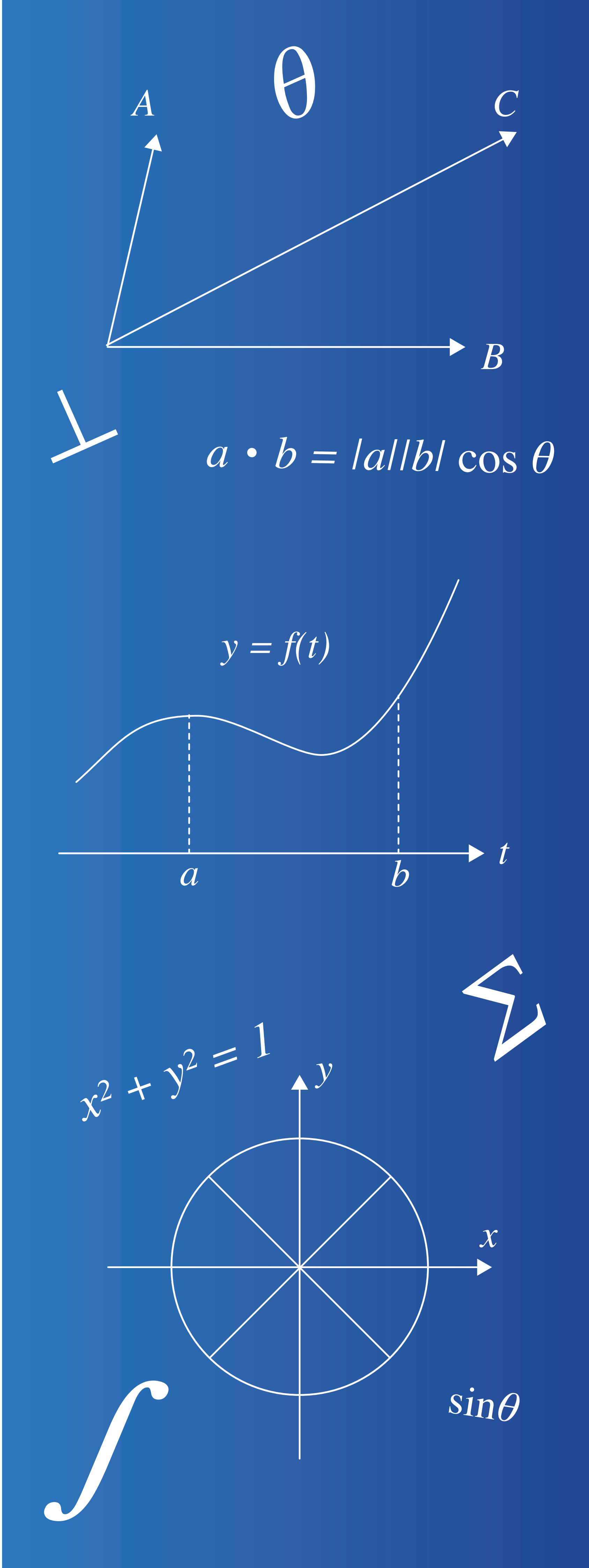

Mathematics

Egyptian Baccalaureate

2nd Year

Advised by
INTERNATIONAL BACCALAUREATE®

The International Baccalaureate™

has reviewed the pedagogical and assessment approaches underpinning this textbook.

Ministry of Education and Technical Education

New Administrative Capital

Cairo, Egypt

Introduction

In alignment with the vision of the Egyptian Baccalaureate System and its commitment to preparing learners for the demands of the twenty-first century, this Mathematics textbook for Grade 11 has been developed to provide students with a rigorous, engaging, and intellectually stimulating learning experience. The curriculum is designed to foster deep conceptual understanding, mathematical literacy, and the ability to apply mathematical knowledge and skills to real-world contexts.

The philosophy underpinning this textbook is grounded in inquiry-based learning, where students actively construct understanding through observation, questioning, investigation, and evidence-based reasoning. Learning begins with meaningful problems, authentic situations, and exploratory experiences that stimulate curiosity and encourage learners to seek explanations for the mathematical patterns, relationships, and structures that occur in the world around them. Through this approach, students are guided to discover patterns, formulate conjectures and explanations, and develop mathematical understanding before engaging with formal concepts, principles, procedures, and models.

The organization of each lesson reflects a coherent progression from exploration to conceptualization and application through the Point-Warm-Up-Explanation-Try-Exercise approach. Students are first introduced to carefully selected contexts, problems, and examples that connect new learning to prior knowledge and everyday experiences. These engaging introductions serve as a catalyst for inquiry, encouraging learners to analyze information, interpret representations, identify patterns, and construct initial explanations. Building upon this exploratory phase, students then engage with the core mathematical concepts, principles, procedures, and models that underpin these situations and provide a structured framework for understanding. The content is intentionally organized to promote deep conceptual understanding rather than rote memorization, enabling learners to recognize relationships among concepts, make meaningful connections between ideas, and apply their knowledge confidently in both familiar and unfamiliar contexts. Through the subsequent Try and Exercise stages, students are provided with opportunities to practice, apply, and assess their understanding, ultimately developing a robust, coherent, and interconnected understanding of mathematics.

This textbook provides extensive opportunities for problem solving, mathematical modeling, and independent investigation. Students are encouraged to test ideas, collect and analyze data, evaluate reasoning, and draw justified conclusions. Through these experiences, learners develop confidence in applying mathematical thinking and become active participants in the construction of knowledge rather than passive recipients of information.

Assessment is integrated throughout the learning process as a tool for both learning and improvement. A variety of formative and summative assessment opportunities are embedded within the lessons to help students monitor their progress, identify areas for growth, and demonstrate their understanding. Assessment tasks are carefully designed to reflect the nature of questions students may encounter in examinations while simultaneously promoting higher-order thinking, analytical reasoning, and the application of knowledge in unfamiliar contexts.

Particular emphasis has been placed on highlighting the Nature of Mathematics as a fundamental component of mathematical literacy. Students are provided with opportunities to understand how mathematical knowledge is developed, justified, refined, and communicated within mathematical communities. The curriculum also encourages appreciation of the contributions of mathematicians and innovators whose ideas and discoveries have shaped our understanding of the mathematical world and contributed to human progress.

The textbook systematically develops mathematical process skills, including problem solving, reasoning, representation, communication, modeling, estimation, pattern recognition, data interpretation, and proof. These skills are integrated within learning experiences to ensure that students not only acquire mathematical knowledge but also develop the competencies required to think and work mathematically.

In keeping with the aspirations of contemporary education, the curriculum promotes global awareness and international-mindedness by connecting mathematical concepts to scientific practices, technological innovations, environmental challenges, and sustainable development goals. Students are encouraged to appreciate the interconnected nature of knowledge and to recognize the role of mathematics in addressing issues that affect communities at local, national, and global levels.

Furthermore, the curriculum seeks to cultivate critical and creative thinking by engaging students in the analysis of information, evaluation of alternative strategies, generation of innovative solutions, and application of mathematical knowledge to authentic challenges. Learners are encouraged to approach problems as mathematicians, scientists, and engineers, employing systematic reasoning and design thinking to investigate questions and develop practical solutions.

A key feature of this textbook is the deliberate integration of mathematics with students' lives and experiences. Throughout the chapters, mathematical ideas are presented within meaningful contexts and supported by contemporary applications drawn from science, engineering, technology, economics, data, sustainability, and everyday life. Such connections reinforce the relevance of mathematics and help learners appreciate its significance in understanding and improving the world around them.

Equally important is the curriculum's commitment to fostering reflective learning. Students are encouraged to engage in self-assessment and metacognitive reflection, enabling them to monitor their understanding, evaluate their learning strategies, and take increasing responsibility for their own academic growth. By reflecting on both what they learn and how they learn, students develop the habits of lifelong learners capable of adapting to new challenges and opportunities.

Ultimately, this textbook aims to develop mathematically literate, responsible, and innovative learners who are equipped with the knowledge, skills, values, and dispositions necessary to thrive in a rapidly changing world. It is our hope that this learning journey will inspire students to explore mathematics with curiosity, think with rigor, act with responsibility, and contribute meaningfully to the advancement of society and the sustainable future of humanity.

MATHEMATICS
Textbook For Grade XI (Part 1)

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MATHEMATICS

Binomial Theorem

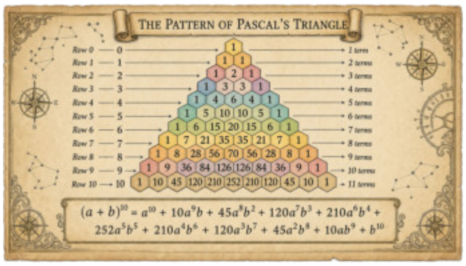
Chapter 1 Algebraic Proofs

Learning Outcomes

Applies the binomial theorem to find the expansion of algebraic expressions and to determine the coefficients of certain terms.

Concept 1 · Binomial and Multinomial Theorems

Consider the first rows of Pascal's triangle, and notice how the number of terms changes in each horizontal row as shown in the figure. If you follow this pattern, how many terms will there be in the tenth row?



Guiding question: How can algebraic formulas and combinatorial coefficients predict the precise coefficient of any term in an expanded polynomial $(x_1 + x_2)^n$ without performing sequential multiplication?

Point!

Calculating ${}_nC_r$

$${}_nC_r = \frac{{}_nP_r}{r!}$$

Example: ${}_5C_3 = \frac{5\times4\times3}{3\times2\times1} = 10$

${}_nC_0 = 1$ ${}_nC_n = 1$

Example: ${}_5C_0 = 1$

Example: ${}_5C_5 = 1$

When expanding $(a+b)^n$, the term with $a^{n-r}\ b^r$ becomes ${}_nC_r\ a^{n-r}\ b^r$.

Exponent of ( )

Exponent of b

Example: When expanding $(a+b)^5$, the term a^3b^2 is

${}_5C_2\ a^3b^2 = \frac{5\times4}{2\times1}a^3b^2 = 10a^3b^2$

Example: When expanding $(x-2y)^6$, the term x^4y^2 is:

Since the exponent of x is 4 and the exponent of $-2y$ is 2, ${}_6C_2 \times x^4 \times (-2y)^2 = \frac{6\times5}{2\times1}\times x^4 \times (-2y)^2 = 15\times x^4\times4\times y^2 = 60x^4y^2$

Expansion of $(a+b)^n$ (Binomial Theorem)

$$(a+b)^n = {}_nC_0\ a^n + {}_nC_1\ a^{n-1}\ b + {}_nC_2\ a^{n-2}\ b^2 + \dots\dots + {}_nC_r\ a^{n-r}\ b^r + \dots\dots + {}_nC_n\ b^n$$

Example: $(x+2)^5 = {}_5C_0 \times x^5 + {}_5C_1 \times x^4 \times 2 + {}_5C_2 \times x^3 \times 2^2 + {}_5C_3 \times x^2 \times 2^3 + {}_5C_4 \times x \times 2^4 + {}_5C_5 \times 2^5$

Start with ${}_5C_0$
Exponent of x : 5
Exponent of 2 : 0

Exponent of x : 4
Exponent of 2 : 1

Exponent of x : 0
Exponent of 2 : 5

Warm Up

Solve the following problems.

1 Find the coefficient of x^4 in the expansion of $(2x^2-3)^7$

2 Find the expansion of $(3x-2)^4$

Explanation

1 In the expansion of $(2x^2-3)^7$, the term x^4 has exponent 2 for $2x^2$, and exponent $7-2=5$ for -3 ${}_7C_5\ (2x^2)^2 \times (-3)^5$

$= \frac{7\times6\times5\times4\times3}{5\times4\times3\times2\times1} \times 4x^4 \times (-243)$

$= -20412x^4$

Therefore, the coefficient of x^4 is $\underline{-20412}$

2 Expand using the Binomial Theorem.

$(3x-2)^4 = {}_4C_0 \times (3x)^4 + {}_4C_1 \times (3x)^3 \times (-2) + {}_4C_2 \times (3x)^2 \times (-2)^2 +$

${}_4C_3 \times 3x \times (-2)^3 + {}_4C_4\ (-2)^4$

$= 1 \times (3x)^4 + \frac{4}{1} \times (3x)^3 \times (-2) + \frac{4\times3}{2\times1} \times (3x)^2 \times (-2)^2 +$

$\frac{4\times3\times2}{3\times2\times1} \times 3x \times (-2)^3 + 1 \times (-2)^4$

$= \underline{81x^4 - 216x^3 + 216x^2 - 96x + 16}$

Try

Solve the following problems.

1 Find the coefficient of the specified term in [ ] for the following expansions.

(a) $(2x-3y)^7[x^6y]$ **(b)** $(3x-2)^5[x^2]$

(c) $(3a^2-2)^8[a^6]$

2 Find the expansion of the following expressions.

(a) $(x+2)^6$ **(b)** $(3a-2b)^5$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the coefficient of the specified term in [ ] for the following expansions.

(a) $(2x-y)^8[x^4y^4]$

(b) $(3a+b)^4[ab^3]$

(c) $(x-4y)^5\ [x^3y^2]$

(d) $(3x+1)^5[x^4]$

(e) $(2a-3)^4[a]$

(f) $(3x+2)^5[x^3]$

(g) $(2x^2-1)^6[x^6]$

(h) $(3x^2-1)^5[x^4]$

(i) $(3x^2+2)^6[x^6]$

2 Find the expansion of the following expressions.

(a) $(x+1)^4$

(b) $(a+b)^4$

(c) $(a-2b)^4$

(d) $(x-2)^5$

(e) $(3x+2y)^5$

(f) $(2a^2-1)^5$

(g) $(x-3)^6$

(h) $(2x-3y)^6$

(i) $\left(x+\frac{1}{3}\right)^6$

★ Challenge — Sum of Coefficients

If $(1+x)^5 = a+bx+cx^2+dx^3+ex^4+fx^5$, then find the numerical value of $a+b+c+d+e+f$.

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MATHEMATICS

Multinomial Theorem

Chapter 1 Algebraic Proofs

If you want to expand an expression consisting of three terms: $(x+y+z)^4$ Without performing the multiplication operation, how can you determine the coefficient of the term containing x^2yz? How do I write without needing to remove the brackets?

Learning Outcomes

Applies the multinomial theorem to find the expansion of algebraic expressions and determine the coefficients of certain terms.

Point!

When expanding $(a+b+c)^n$, the term containing $a^p b^q c^r$ is:

$$\frac{n!}{p!q!r!}a^p b^q c^r$$

Warm Up

Find the coefficient of the term a^4b^3c in the following expansions.

(a) $(a+b+c)^8$

(b) $(a-2b+3c)^8$

Explanation

(a) When expanding $(a+b+c)^8$, the term a^4b^3c is

$$\frac{8!}{4!3!1!}a^4b^3c^1$$
$$= 280a^4b^3c$$

Therefore, the required coefficient is $\underline{280}$

(b) When expanding $(a-2b+3c)^8$, the term a^4b^3c is

$$\frac{8!}{4!3!1!}a^4\times(-2b)^3\times(3c)^1$$
$$= -6720a^4b^3c$$

Therefore, the required coefficient is $\underline{-6720}$

Try

Solve the following problems.

1 Find the coefficient of the following terms in the expansion of $(a+b+c)^6$

(a) a^3bc^2

(b) $a^2b^2c^2$

2 Find the coefficient of the following terms in the expansion of $(a+3b-2c)^6$

(a) $a^2b^2c^2$

(b) ab^2c^3

Work with a partner

Collaborate with your group to answer question 2, then discuss the answer.

Exercise

Solve the following problems.

1 Find the coefficient of the following terms in the expansion of $(x+y+z)^7$

(a) xy^5z

(b) x^4y^2z

(c) $x^2y^3z^2$

2 Find the coefficient of the following terms in the expansion of $(x+y+z)^9$

(a) x^6yz^2

(b) $x^3y^3z^3$

(c) $x^2y^4z^3$

3 Find the coefficient of the following terms in the expansion of $(a+2b-3c)^7$

(a) $a^2b^3c^2$

(b) abc^5

(c) a^2b^4c

4 Find the coefficient of the following terms in the expansion of $(2a-b+3c)^8$

(a) $a^2b^2c^4$

(b) a^4bc^3

(c) $a^3b^3c^2$

★ Challenge — Greatest Coefficient

Find the greatest coefficient in the expansion of $\left(1+x+x^2\right)^4$

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Applications of the Binomial Theorem

Chapter 1 Algebraic Proofs

Point!

For proof problems using the Binomial Theorem, use the expansion of $(1+x)^n$

$(1+x)^n = {}_nC_0 + {}_nC_1\,x + {}_nC_2\,x^2 + \ldots\ldots + {}_nC_n\,x^n$

Substitute an appropriate value for x to derive the required equality.

Learning Outcomes

Employs the binomial theorem in mathematical and computational applications.

Warm Up

Prove that the following equality holds.

${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots\ldots + {}_nC_n = 2^n$

Explanation

[Proof]

Expanding $(1+x)^n$ using the Binomial Theorem:

$(1+x)^n = {}_nC_0 + {}_nC_1\,x + {}_nC_2\,x^2 + \ldots\ldots + {}_nC_n\,x^n$

Substitute $x = 1$ to make $(1+x)^n = 2^n$

Substituting $x = 1$ into this expression,

$2^n = {}_nC_0 + 1 \times {}_nC_1 + 1^2 \times {}_nC_2 + \ldots\ldots + 1^n \times {}_nC_n$

$2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots\ldots + {}_nC_n$

Therefore, the given equality holds.

Try

Prove that the following equality holds.

${}_nC_0 - {}_nC_1 + {}_nC_2 - \ldots\ldots + (-1)^n\,{}_nC_n = 0$

Work with a partner

Collaborate with your group to answer the try section then discuss the answer.

Exercise

Prove that the following equalities hold.

1. ${}_nC_0 + 2\,{}_nC_1 + 2^2\,{}_nC_2 + \ldots\ldots + 2^n\,{}_nC_n = 3^n$
2. ${}_nC_0 - 2\,{}_nC_1 + 2^2\,{}_nC_2 - \ldots\ldots + (-2)^n\,{}_nC_n = (-1)^n$
3. ${}_nC_0 - \dfrac{{}_nC_1}{2} + \dfrac{{}_nC_2}{2^2} - \ldots\ldots + (-1)^n\dfrac{{}_nC_n}{2^n} = \left(\dfrac{1}{2}\right)^n$

★ Challenge — Divisibility

By using binomial theorem: Prove that $(8^n - 1)$ is divisible by 7 (n is a positive integer).

Reflect

Compare binomial expansion and multinomial expansion. In what ways are they similar, and how does seeing the binomial as a special case of the multinomial help you unify your understanding of algebraic powers and combinatorial counting methods?

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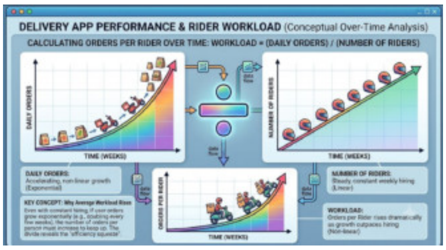
MATHEMATICS

Division of Polynomials

Chapter 1 Algebraic Proofs

Concept 2 · Polynomial Division

When you were managing a smart food delivery app, you would notice that the number of daily orders grows at an accelerating, non-linear rate as the app becomes more popular. At the same time, the app hires new delivery riders at a steady, constant weekly rate (a linear rate). To find out the average workload—or "how many orders a single rider needs to cover" over time—you have to divide the large function of total orders by the function of the number of riders.



This is exactly where polynomial division comes into play. It is the mathematical tool we use when we want to break down a large, complex, rapidly growing system and distribute its load into smaller units to find the exact share or efficiency of a single part.

Guiding Question: How does dividing complex algebraic expressions help us simplify massive amounts of data and understand the share of a single unit within a constantly growing system?

Point!

Division of polynomials can be calculated using long division, similar to integers.

In problems where you "treat as a polynomial in x", **treat other variables as constants**.

Warm Up

Solve the following problems.

1 Find the quotient and the remainder when $2x^3-5x^2-2$ is divided by $2x^2-x-1$

2 For polynomials $A=4x^2-9ax-9a^2$ and $B=x-3a$, treating each as a polynomial in x, find the quotient and the remainder when A is divided by B

Explanation

1

$$
\begin{array}{r|l}
 & x-2 \\ \cline{2-2}
2x^2-x-1 & 2x^3-5x^2-2 \\
 & 2x^3-x^2-x \\ \cline{2-2}
 & -4x^2+x-2 \\
 & -4x^2+2x+2 \\ \cline{2-2}
 & -x-4
\end{array}
$$

Therefore, **the quotient is $x-2$ and the remainder is $-x-4$**

Learning Outcomes

Performs polynomial division using standard algorithm.

Common Mistakes

Errors in subtraction: Mistakes often happen when subtracting after multiplication, especially forgetting to change signs or combine like terms properly. Partial multiplication: Multiplying only by the leading term of the divisor instead of the entire divisor.

If a term of a certain degree is missing in the dividend, leave that space blank.

Subtract $(2x^2-x-1)\times x$ to eliminate $2x^3$

Subtract $(2x^2-x-1)\times(-2)$ to eliminate $-4x^2$

It is the remainder when the degree becomes lower than the divisor.

2 Since we treat it as a polynomial in x, treat variables other than x as numbers.

$$\begin{array}{r|l} & 4x + 3a \\ \cline{2-2} x - 3a & 4x^2 - 9ax - 9a^2 \\ & 4x^2 - 12ax \\ \cline{2-2} & \quad 3ax - 9a^2 \\ & \quad 3ax - 9a^2 \\ \cline{2-2} & \qquad 0 \end{array}$$

Therefore, **the quotient is $4x + 3a$ and the remainder is 0**

Try

Find the quotient and the remainder when polynomial A is divided by polynomial B.(For **2**, calculate treating it as a polynomial in x).

1 $A = 2x^3 - 13x + 1, B = x^2 + 2x - 3$

2 $A = x^3 - 3ax^2 + 5a^2x - 5a^3,\ B = x - 2a$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the quotient and the remainder when polynomial A is divided by polynomial B.(For **7–10**, calculate treating it as a polynomial in x).

1 $A = x^2 - 5x + 1, B = x - 2$

2 $A = x^3 - 4x + 1, B = x - 3$

3 $A = 2x^3 - 3x^2 + 3x + 4, B = x^2 - 2x + 3$

4 $A = 6x^3 - 8x^2 + 5, B = 2x^2 - 2x - 5$

5 $A = x^3 - 7x + 6, B = x^2 + 2x - 3$

6 $A = 3x^4 - 5x^3 + x - 1, B= -x^2 + 3x + 2$

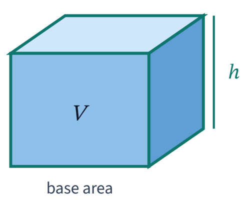
7 $A = 4x^2 - 5ax - 6a^2,\ B = x - 2a$

8 $A = x3 + ax2 + a2x - a3,\ B = x + 2a$

9 $A = x^3 - 2ax^2 + 3a^3,\ B = x + a$

10 $A = x^3 + 3x^2y - 2y^3,\ B = x + y$

★ Challenge — Find the Height



A cuboid shaped box of volume $4x^3 + 4x^2 - 9x - 9$. If the area of its base is $2x^2 - x - 3$, write its height in terms of x

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MATHEMATICS

Quotient and Remainder of Polynomials

Chapter 1 Algebraic Proofs

Point!

If a polynomial A is divided by a polynomial B, yielding a quotient Q and a remainder R, then the following equality holds:

$A = B \times Q + R$

Example: When 7 is divided by 3, the quotient is 2 and remainder is 1,

$7 = 3 \times 2 + 1$

In problems where the quotient and the remainder are given, substitute the known expressions into $A = BQ + R$.

Learning Outcomes

Distinguishes the quotient and the remainder in polynomial division, deduces that the degree of the remainder is less than the degree of the divisor, and uses this fact to verify the correctness of the division.

Warm Up

Find the polynomials A and B that satisfy the following conditions.

1 When A is divided by $x^2 - 2x - 1$, the quotient is $x + 3$ and remainder is $-2x + 5$

2 When $4x^3 + 8x - 9$ is divided by B, the quotient is $2x - 3$ and remainder is $x + 15$

Common Mistakes

Stopping too early: Ending the division before reaching a remainder of lower degree than the divisor Confusing quotient and remainder: Thinking the remainder must always be zero, while in fact it can be a term (or more) of lower degree than the divisor.

Explanation

Since the quotient and the remainder are given, substitute the known expressions into $A = BQ + R$.

1 $A = \left(x^2 - 2x - 1\right)(x + 3) + (-2x + 5)$

$= x^3 + 3x^2 - 2x^2 - 6x - x - 3 - 2x + 5$

$= x^3 + x^2 - 9x + 2$

2 $4x^3 + 8x - 9 = \mathrm{B} \times (2x - 3) + (x + 15)$

$B \times (2x - 3) + (x + 15) = 4x^3 + 8x - 9$

$B \times (2x - 3) = 4x^3 + 7x - 24$

$B = \left(4x^3 + 7x - 24\right) \div (2x - 3)$

$= 2x^2 + 3x + 8$

Solve for B

$$\begin{array}{r|l}
 & 2x^2 + 3x + 8 \\
\hline
2x - 3 & 4x^3 \qquad + 7x - 24 \\
 & \underline{4x^3 - 6x^2} \\
 & \qquad 6x^2 + 7x \\
 & \qquad \underline{6x^2 - 9x} \\
 & \qquad\qquad 16x - 24 \\
 & \qquad\qquad \underline{16x - 24} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Remainder must be 0

Try

Find the polynomials A and B that satisfy the following conditions.

1. When A is divided by $2x^2 + x + 1$, the quotient is $3x + 2$ and the remainder is $-2x + 6$
2. When $x^3 - x^2 + 3x + 1$ is divided by B, the quotient is $x^2 - 2x + 5$ and the remainder is -4

Work with a partner

Collaborate with your group to solve question 2, then discuss the answer.

Exercise

Find the polynomials A and B that satisfy the following conditions.

1. When A is divided by $x^2 - 2x - 1$, the quotient is $2x - 3$ and the remainder is $-2x$.
2. When A is divided by $x^2 - 3x + 2$, the quotient is $2x + 6$ and the remainder is $2x - 3$
3. When A is divided by $2x^2 - x + 2$, the quotient is $2x - 1$ and the remainder is 2
4. When $2x^3 + x^2 - 3x + 5$ is divided by B, the quotient is $x + 1$ and the remainder is $-5x + 2$
5. When $6x^3 + 11x^2 - 2$ is divided by B, the quotient is $2x^2 + 3x - 1$ and the remainder is -1
6. When $2x^3 + 4x^2 + 7$ is divided by B, the quotient is $x + 2$ and the remainder is $3x + 13$

★ Challenge — Recover the Divisor

When $x^4 + x^2 + 3x - 1$ is divided by B, the quotient is $x^2 + x + 1$ with the remainder $3x - 2$, then find the polynomial B.

Reflect

When we divide one polynomial by another, why must the remainder always have a degree less than the divisor? How does this condition help us verify the correctness of the division steps and understand the relationship between the quotient and the remainder?

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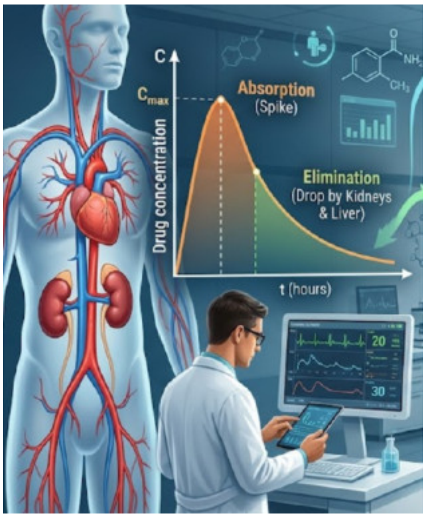
MATHEMATICS

Multiplication and Division of Rational Expressions

Chapter 1 Algebraic Proofs

Concept 3 · Rational Expressions

When a patient takes a prescription medication, the concentration of the drug in their bloodstream changes rapidly over time. Medical researchers track this concentration using math. As the body absorbs the drug, the amount in the blood spikes, but as the kidneys and liver filter it out, the concentration drops.



To model this lifecycle accurately, scientists use rational expressions—fractions that have algebraic expressions in both the numerator and the denominator. When doctors need to combine treatments, alter dosages, or calculate how fast two different drugs clear out of a patient's system simultaneously, they must use the four main arithmetic operations (addition, subtraction, multiplication, and division) on these algebraic fractions.

Guiding Question: How do the fundamental laws of arithmetic allow us to combine and simplify complex, shifting algebraic fractions to predict outcomes in medicine and life sciences?

Learning Outcomes

Simplifies fractional (rational) expressions by doing multiplication and division

Point!

Dividing both the numerator and denominator of a rational expression by their **common factor** is called **reducing**.

A rational expression that cannot be reduced further is called an **irreducible rational expression**.

Example: Reducing $\dfrac{(x-1)(x-3)}{x(x-3)}$ gives $\dfrac{(x-1)\cancel{(x-3)}}{x\cancel{(x-3)}} = \dfrac{x-1}{x}$

Irreducible rational expression since it cannot be simplified further.

When there are polynomials in the denominator or numerator of a rational expression, first **factorise** them to check if they can be reduced.

Warm Up

Solve the following problems.

1 Simplify the following rational expressions.

(a) $\dfrac{12ab^5}{9a^3b^2}$ (b) $\dfrac{x^2-4}{3x^2+7x+2}$

2 Calculate the following.

$\dfrac{x^2-4x-5}{2x^2+5x+2} \div \dfrac{x^2-x-20}{2x^2-5x-3}$

Explanation

1

(a) $\dfrac{{}^{4}\cancel{12}\cancel{a}\cancel{b^5}{}^{3}}{{}^{3}\cancel{9}\cancel{a^3}{}^{2}\,\cancel{b^2}}$

$= \dfrac{4b^3}{3a^2}$

(b) $\dfrac{x^2-4}{3x^2+7x+2}$ Factorise.

$= \dfrac{\cancel{(x+2)}(x-2)}{(3x+1)\cancel{(x+2)}}$ Cancel the common factor $(x+2)$.

$= \dfrac{x-2}{3x+1}$

2 $\dfrac{x^2-4x-5}{2x^2+5x+2} \div \dfrac{x^2-x-20}{2x^2-5x-3}$ Change division to multiplication by using the reciprocal of the divisor.

$= \dfrac{x^2-4x-5}{2x^2+5x+2} \times \dfrac{2x^2-5x-3}{x^2-x-20}$ Factorise.

$= \dfrac{(x+1)\cancel{(x-5)}}{\cancel{(2x+1)}(x+2)} \times \dfrac{\cancel{(2x+1)}(x-3)}{\cancel{(x-5)}(x+4)}$ Cancel out common factors.

$= \dfrac{(x+1)(x-3)}{(x+2)(x+4)}$ Do not expand the brackets.

Common Mistakes

Students often mistakenly cross out identical variables or numbers in the numerator and denominator, even when they are separated by addition or subtraction signs.

Try

Solve the following problems.

1 Simplify the following rational expressions.

(a) $\dfrac{6x^2y^3}{20x^3y}$ (b) $\dfrac{2x^2-3x-5}{x^2+3x+2}$

2 Calculate the following.

(a) $\dfrac{x^2-11x+24}{x^2-6x-16} \times \dfrac{x^2+2x}{x^2-6x+9}$

(b) $\dfrac{x^2-9}{x^2-x-2} \div \dfrac{2x^2+5x-3}{x+1}$

Work with a partner

Collaborate with your group to solve question 2, then discuss the answer.

Exercise

Solve the following problems.

1 Simplify the following rational expressions.

(a) $\dfrac{6a^2bc^2}{8ab^4c^2}$

(b) $\dfrac{2a^2b}{8a^3b^3}$

(c) $\dfrac{x^2+3x}{x^2-9}$

(d) $\dfrac{x^2+x-2}{x^2-5x+4}$

(e) $\dfrac{2x^2-4xy+2y^2}{x^3+3x^2y-4xy^2}$

(f) $\dfrac{x^3-1}{x^2+2x-3}$

2 Calculate the following.

(a) $\dfrac{a^3b^2}{3c}\times\dfrac{9c^3}{a^4b^2}$

(b) $\dfrac{6bc^3}{5a^2}\times\dfrac{10a^4}{3b^3}$

(c) $\dfrac{2xy}{3ab}\div\dfrac{8x^3y}{9a^2b^3}$

(d) $\dfrac{x+1}{x^2-4}\times\dfrac{x^2-2x-8}{x^2+x}$

(e) $\dfrac{x^2-9}{x^2-2x}\times\dfrac{x^2-7x+10}{x^2+x-12}$

(f) $\dfrac{x^2+x-6}{2x^2+x-6}\times\dfrac{x^2+6x+8}{3x^2-5x-2}$

(g) $\dfrac{2x^2-x-10}{x^2-2x-3}\div\dfrac{x^2+x-2}{x^2-6x+9}$

(h) $\dfrac{x^2-3x-10}{2x^2+3x-2}\div\dfrac{x^2-2x-15}{6x-3}$

(i) $\dfrac{x^2-y^2}{x^2-2xy+y^2}\div\dfrac{x^2+xy}{x-y}$

★ Challenge — Simplify Then Evaluate

If $x=\sqrt{123}$, find the value of $\dfrac{x^3+8}{x^2-x-6}\times\dfrac{x^2-6x+9}{x^2-2x+4}\div\dfrac{2x-6}{x^5-1}$

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MATHEMATICS

Addition and Subtraction of Rational Expressions

Chapter 1 Algebraic Proofs

Learning Outcomes

Simplifies fractional (rational) expressions by doing addition and subtraction

Point!

Addition and subtraction of rational expressions with different denominators

Just like with numerical fractions, multiply the numerator and denominator by the same expression to find a common denominator before calculating.

Example: $\frac{1}{x+1}+\frac{1}{x-2}=\frac{(x-2)}{(x+1)(x-2)}+\frac{(x+1)}{(x-2)(x+1)}$

$=\frac{(x-2)+(x+1)}{(x+1)(x-2)}=\frac{2x-1}{(x+1)(x-2)}$

When finding a common denominator, first **factorise** the denominators.

Example: $\frac{2}{x^2-x-2}+\frac{3}{x^2-5x+6}=\frac{2}{(x+1)(x-2)}+\frac{3}{(x-2)(x-3)}$

$=\frac{2(x-3)}{(x+1)(x-2)(x-3)}+\frac{3(x+1)}{(x-2)(x-3)(x+1)}$

$=\frac{2(x-3)+3(x+1)}{(x+1)(x-2)(x-3)}=\frac{5x-3}{(x+1)(x-2)(x-3)}$

After combining over a common denominator, calculate only the numerator, and check if it can be **factorised** and **reduced** .

Common Mistakes

Adding or subtracting Numerators and Denominators Directly: The Error: Students often treat rational expressions like regular integers, adding or subtracting the numerators together and the denominators together.

Warm Up

Simplify the following expressions.

$$\frac{x+4}{x^2-2x}-\frac{3}{x^2-3x+2}$$

Explanation

$$\frac{x+4}{x^2-2x}-\frac{3}{x^2-3x+2}$$

First, factorise the denominator.

$$=\frac{x+4}{x(x-2)}-\frac{3}{(x-1)(x-2)}$$

Find a common denominator.

$$=\frac{(x+4)(x-1)-3x}{x(x-1)(x-2)}$$

Calculate only the numerator.

$$=\frac{x^2-4}{x(x-1)(x-2)}$$

Factorise the numerator.

$$=\frac{\cancel{(x-2)}(x+2)}{x(x-1)\cancel{(x-2)}}$$

Always reduce the fraction (cancel common factors).

$$=\frac{x+2}{x(x-1)}$$

Do not expand the denominator.

Try

Simplify the following expressions.

1. $\dfrac{x-1}{x^2+4x+3}+\dfrac{2}{x^2+5x+6}$

2. $\dfrac{x}{x-1}-\dfrac{2}{x^2-1}$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Simplify the following expressions.

1. $\dfrac{x^2-4}{x+1}+\dfrac{3}{x+1}$

2. $\dfrac{3}{x^2+x}-\dfrac{3-x}{x^2+x}$

3. $\dfrac{x+2}{x}-\dfrac{x-2}{x-1}$

4. $\dfrac{1}{x+2}+\dfrac{1}{x-2}$

5. $\dfrac{2}{x-1}+\dfrac{3}{x^2+5x-6}$

6. $\dfrac{x-3}{x^2+x}+\dfrac{x}{x^2-1}$

7. $\dfrac{4x}{x^2-1}-\dfrac{x-1}{x^2+x}$

8. $\dfrac{1}{x^2+5x+6}+\dfrac{1}{x^2+7x+12}$

9. $\dfrac{x-1}{x^2+3x+2}-\dfrac{x-3}{x^2+4x+3}$

10. $\dfrac{x+4}{x^2+2x}-\dfrac{x+5}{x^2+x-2}$

11. $\dfrac{x+8}{x^2+x-2}-\dfrac{x+5}{x^2-1}$

★ Challenge — Two-Stage Water Filtration

An environmental engineering team is monitoring a dual-stage water filtration system at a community reservoir. The system extracts organic pollutants over a timeline of x hours. Stage 1 (Primary Bio-Filter): the baseline rate of pollutant extraction is $\dfrac{x^3+8x}{x^2-x-2}$. Stage 2 (Secondary Charcoal Filter): a small amount of backflow re-enters, reducing the cleanup rate, modelled by $\dfrac{8}{x-2}$. To determine the net purification efficiency, subtract the Stage 2 backflow rate from the Stage 1 extraction rate.

1-8
MATHEMATICS

Compound Fractions

Chapter 1 Algebraic Proofs

Point!

Expressions that contain rational expressions in the denominator or numerator, like the one on the right, are called **compound fractions**.

$$\frac{\frac{1}{y}}{\frac{2}{x}},\quad \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

When calculating compound fractions,

multiply the numerator and denominator by the same expression.

Learning Outcomes

Simplifies compound fractions and converts them to their simplest form.

Warm Up

Simplify the following expressions.

1 $\dfrac{1}{1-\dfrac{1}{x-1}}$

2 $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{1}{1-x}-\dfrac{1}{1+x}}$

Explanation

1

$$\frac{1}{1-\frac{1}{x-1}}$$

$$=\frac{1\times(x-1)}{\left(1-\frac{1}{x-1}\right)\times(x-1)}$$

$$=\frac{x-1}{(x-1)-\frac{\cancel{x-1}}{\cancel{x-1}}}$$

$$=\frac{x-1}{(x-1)-1}$$

$$=\frac{x-1}{x-2}$$

Multiply numerator and denominator by $x-1$

2

$$\frac{\frac{1}{1-x}+\frac{1}{1+x}}{\frac{1}{1-x}-\frac{1}{1+x}}$$

$$=\frac{\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\times(1-x)(1+x)}{\left(\frac{1}{1-x}-\frac{1}{1+x}\right)\times(1-x)(1+x)}$$

$$=\frac{\frac{\cancel{(1-x)}(1+x)}{\cancel{1-x}}+\frac{(1-x)\cancel{(1+x)}}{\cancel{1+x}}}{\frac{\cancel{(1-x)}(1+x)}{\cancel{1-x}}-\frac{(1-x)\cancel{(1+x)}}{\cancel{1+x}}}$$

$$=\frac{(1+x)+(1-x)}{(1+x)-(1-x)}$$

$$=\frac{2}{2x}$$

$$=\frac{1}{x}$$

Multiply numerator and denominator by $(1-x)(1+x)$

Try

Simplify the following expressions.

1 $\dfrac{1}{1-\dfrac{x}{x+1}}$

2 $\dfrac{\dfrac{1}{x+2}+\dfrac{1}{x-2}}{\dfrac{1}{x+2}-\dfrac{1}{x-2}}$

3 $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$

> 👥 **Work with a partner**
>
> Collaborate with your group to solve question 3, then discuss the answer.

Exercise

Simplify the following expressions.

1 $\dfrac{x-\dfrac{2}{x-1}}{\dfrac{x}{x-1}-2}$

2 $\dfrac{1+\dfrac{x-y}{x+y}}{1-\dfrac{x-y}{x+y}}$

3 $\dfrac{\dfrac{1}{x+y}+\dfrac{1}{x-y}}{\dfrac{1}{x+y}-\dfrac{1}{x-y}}$

4 $\dfrac{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}$

5 $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$

★ Challenge — Signal Booster

In telecommunications, when a signal travels through a series of repeating booster nodes, its strength degrades recursively. A network engineer models the efficiency $E(x)$ of a 3-stage booster using the following compound fraction:

$$E(x)=\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}$$

where $x>0$. Convert this compound fraction into a single rational expression $\dfrac{A(x)}{B(x)}$.

Reflect

Consider the following mathematical statement: $\dfrac{x^2-16}{x-4}=x+4$ Is this equation an absolute identity for all real numbers x? Explain why or why not. If a student graphs $y=\dfrac{x^2-16}{x-4}=x+4$ on a graphing calculator, it appears identical to the line $y=x+4$. What specific feature of the graph is hidden by typical screen resolutions at $x=4$?

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MATHEMATICS

Identities

Chapter 1 Algebraic Proofs

Concept 4 · Identities and Proofs

If you have ever worn a fitness tracker that monitors your heart rate, or looked at an ECG (electrocardiogram) wave monitor in a medical drama, you have seen complex signal processing in action. Biological rhythms—like the beating of a human heart or the electrical firing of neurons—are cyclical. Medical devices track these rhythms using layered, complex wave equations.

If a pacemaker or medical monitor had to calculate these massive equations in their rawest, bulkiest forms every millisecond, the device's battery would drain rapidly, risking the patient's life. To prevent this, software engineers replace massive wave formulas with streamlined, simplified equivalents. This substitution is only possible because of mathematical identities—absolute laws showing that two entirely different-looking expressions yield the exact same output for every single value. Proving equalities isn't about solving for an unknown variable; it is about proving that two different mathematical pathways are fundamentally the same.

Guiding Question: How do we mathematically verify that two distinct expressions are identically equal for all domain values, and how does manipulating one side of an equation preserve structural equivalence without altering its output?

Learning Outcomes

Recognises an algebraic identity and distinguishes it from an equation.

Point!

An **equality that holds true for any value substituted** for x is called an **identity** in x.

$ax^2 + bx + c = a'x^2 + b'x + c'$ is an identity in x.

$\Leftrightarrow$ **$a = a'$, $b = b'$, $c = c'$**

Coefficients of terms of the same degree are equal on both sides.

When solving identity problems in x, arrange the given expression in **descending order of powers of x**, then compare the coefficients of each term.

For identities involving rational expressions, first clear the denominators.

Warm Up

Find the values of the constants $a,\ b$, and c such that the following equations are identities in x.

1 $x^2+6=a(x+1)^2+b(x+1)+c$

2 $\dfrac{3x+1}{(x+2)(x-3)}=\dfrac{a}{x+2}+\dfrac{b}{x-3}$

Explanation

1 $x^2+6=a(x+1)^2+b(x+1)+c$

$x^2+6=ax^2+(2a+b)x+(a+b+c)$

Therefore, $1=a,\ 0=2a+b,\ 6=a+b+c$

Solving this system of equations gives $a=1$, $b=-2$, and $c=7$

Rearrange the RHS in descending powers of x.

Compare the coefficients of terms with the same degree.

- If there is no x term, consider the coefficient to be 0
- Solve the simultaneous equations for $a,\ b$, and c.

2 $\dfrac{3x+1}{(x+2)(x-3)}=\dfrac{a}{x+2}+\dfrac{b}{x-3}$

$3x+1=a(x-3)+b(x+2)$

$3x+1=(a+b)x+(-3a+2b)$

Therefore, $3=a+b,\ 1=-3a+2b$

Solving this system of equations gives $a=1$ and $b=2$

Multiply both sides by $(x+2)(x-3)$ to clear the denominators.

Rearrange the RHS in descending powers of x.

Try

Find the values of the constants $a,\ b$, and c such that the following equations are identities in x.

1 $x^2-x-3=a(x-1)^2+b(x-1)+c$

2 $\dfrac{1}{(2x-1)(x+1)}=\dfrac{a}{2x-1}+\dfrac{b}{x+1}$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the values of the constants a, b, and c such that the following equations are identities in x.

1. $a(x+1) - b(x+5) = 8x$

2. $(x+a)(b+3) = 4x+12$

3. $2x^2 + 5x + 4 = (x+1)(ax+b) + c$

4. $(ax+b)(x+1) = 4x^2 + cx + 1$

5. $2x^2 - 7x + 8 = (x-3)(ax+b) + c$

6. $\dfrac{a}{x-1} + \dfrac{b}{x-5} = \dfrac{5x-1}{(x-1)(x-5)}$

7. $\dfrac{x+8}{(x-2)(x+3)} = \dfrac{a}{x-2} + \dfrac{b}{x+3}$

8. $\dfrac{3x-8}{3x^2+5x-2} = \dfrac{a}{x+2} + \dfrac{b}{3x-1}$

★ Challenge — Partial Fractions

An environmental toxicologist is studying how a specific organic compound clears from a water filtration ecosystem. The clearance rate efficiency $f(t)$ depends on time t (hours, $t > 0$) and is modelled by

$$f(t) = \frac{22t + 15}{8t^2 + 10t + 3}$$

Separate it into two partial fractions $\dfrac{a}{2t+1} + \dfrac{b}{4t+3}$, then find a and b.

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MATHEMATICS

Proving Equalities (1)

Chapter 1 Algebraic Proofs

Learning Outcomes

Prove the validity of algebraic equalities and identities in regular cases (without any prerequisites)

Point!

To prove the equality LHS=RHS, simplify the LHS and RHS respectively so that they become the **same expression** .

Warm Up

Prove the following equalities.

1 $(x+y)^2-(x-y)^2=4xy$

2 $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$

Explanation

1 Calculate and simplify the LHS to derive the same expression as the RHS.

[Proof]

$\text{LHS}=(x^2+2xy+y^2)-(x^2-2xy+y^2)$ — Start with the LHS

$=x^2+2xy+y^2-x^2+2xy-y^2$

$=4xy$ — The same expression as the RHS

Therefore, $(x+y)^2-(x-y)^2=4xy$

2 Calculate and simplify the LHS and RHS respectively to derive the same expression.

[Proof]

$\text{LHS}=a^2c^2+a^2d^2+b^2c^2+b^2d^2 \ldots\ldots \text{(i)}$ — Start with the LHS

$\text{RHS}=(a^2c^2-2abcd+b^2d^2)+(a^2d^2+2abcd+b^2c^2)$

$=a^2c^2+b^2d^2+a^2d^2+b^2c^2$

$=a^2c^2+a^2d^2+b^2c^2+b^2d^2 \ldots\ldots \text{(ii)}$ — Rearrange terms to match (i)

From (i) and (ii), $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$

Try

Prove the following equalities.

1. $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2$
2. $\left(a^2-b^2\right)\left(c^2-d^2\right) = (ac-bd)^2 - (ad-bc)^2$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following equalities.

1. $\left(x^2+x+1\right)\left(x^2-x+1\right) = x^4 + x^2 + 1$
2. $(a+b)^3 - 3ab(a+b) = a^3 + b^3$
3. $(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) = a^3+b^3+c^3-3abc$
4. $(a+2b)^2 - 4ab = (a-2b)^2 + 4ab$
5. $\left(x^2+1\right)\left(y^2+1\right) = (xy+1)^2 + (x-y)^2$
6. $(2a+b)^2 - (a+2b)^2 = 3\left(a^2-b^2\right)$

★ Challenge

Prove that: $\left(a^2-ab+b^2\right)\left(a^2+ab+b^2\right) = \left(a^2+b^2\right)^2 - (ab)^2$

1-11

MATHEMATICS

Proving Equalities (2)

Chapter 1 Algebraic Proofs

Point!

In proving conditional equalities, **express one variable from the conditional equation in terms of the others** and substitute it to eliminate that variable.

Example: If $x + y = 1$, substitute $y = 1 - x$ to eliminate y.

Learning Outcomes

Prove the validity of algebraic equalities and identities in conditional cases

Warm Up

If $a + b + c = 0$, prove the following equality.

$a^2 + b^2 + c^2 = -2(ab + bc + ca)$

Explanation

[Proof]

From $a + b + c = 0$, $c = -a - b$...... (i)

Substituting (i) into the LHS,

$\text{LHS} = a^2 + b^2 + (-a - b)^2$

$= a^2 + b^2 + a^2 + 2ab + b^2$

$= 2a^2 + 2ab + 2b^2$ ...... (ii)

Substituting (i) into the RHS,

$\text{RHS} = -2\{ab + b(-a - b) + (-a - b)a\}$

$= -2\left(ab - \text{ab} - b^2 - a^2 - ab\right)$

$= -2\left(-a^2 - ab - b^2\right)$

$= 2a^2 + 2ab + 2b^2$ ...... (iii)

From (ii) and (iii), $a^2 + b^2 + c^2 = -2(ab + bc + ca)$

Express one variable in the conditional equation in terms of the others

Substitute (i) to eliminate c

Try

Prove the following equalities.

1 If $x + y = 1$, $x^2 + y^2 + 1 = 2(x + y - xy)$

2 If $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following equalities.

1. If $x+y=1, x^2+y^2=1-2xy$
2. If $x-y=1, x^2+y^2=2xy+1$
3. If$x+y=1, x^2-x=y^2-y$
4. If $a+b+c=0, a^2+ca=b^2+bc$
5. If $a+b+c=0,\ (a+b)(b+c)(c+a)+abc=0$

★ Challenge — A Constant Sum

Three non-zero numbers $a,\ b,\ c$ satisfy $a+b+c=0$. Prove that, whichever such numbers you choose, the sum below always equals exactly -3 :

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=-3$$

Reflect

Why is it mathematically invalid to perform cross-multiplication or add terms to both sides when proving an identity, and how does working strictly on one side ensure that the proof remains logically sound?

1-12

MATHEMATICS

Proving Equalities (3)

Chapter 1 Algebraic Proofs

Point!

An equation expressing that ratios or values of ratios are equal, like

$\frac{a}{b} = \frac{c}{d}$ or $x : y : z = a : b : c$, is called a **proportion**.

For problems with condition $\frac{a}{b} = \frac{c}{d}$, first let $\frac{a}{b} = \frac{c}{d} = k$ and express as

$a = bk, c = dk$

For problems with condition $x : y : z = a : b : c$, let **k be a constant** and

express as $x = ak, y = bk, z = ck$

Learning Outcomes

Prove the validity of algebraic equalities and identities in proportional cases.

Warm Up

Solve the following problems.

1 If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{ab}{a^2 + b^2} = \frac{cd}{c^2 + d^2}$

2 If $x : y : z = 2 : 3 : 5$, find the value of $\frac{2xy + yz + 3zx}{x^2 + y^2 + z^2}$

Explanation

1 [Proof]

Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, then $a = bk,\ c = dk$...... (i)

Substituting (i) into the LHS,

Eliminate a

$$\text{LHS} = \frac{bk \times b}{(\text{bk})^2 + b^2}$$

$$= \frac{b^2k}{b^2k^2 + b^2}$$

Factorise and simplify

$$= \frac{b^2k}{b^2(k^2 + 1)}$$

$$= \frac{k}{k^2 + 1} \text{ ...... (ii)}$$

Substituting (i) into the RHS,

Eliminate c

$$\text{RHS} = \frac{dk \times d}{(\text{dk})^2 + d^2}$$

$$= \frac{d^2k}{d^2k^2 + d^2}$$

Factorise and simplify

$$= \frac{d^2k}{d^2(k^2 + 1)}$$

$$= \frac{k}{k^2 + 1}$$

From (ii) and (iii), $\dfrac{ab}{a^2 + b^2} = \dfrac{cd}{c^2 + d^2}$

2 From $x : y : z = 2 : 3 : 5$, let k be a constant,

$x = 2k,\ y = 3k,\ z = 5k$...... (i)

from (i), $\dfrac{2xy + yz + 3zx}{x^2 + y^2 + z^2} = \dfrac{2 \times 2k \times 3k + 3k \times 5k + 3 \times 5k \times 2k}{(2k)^2 + (3k)^2 + (5k)^2}$

$$= \frac{57k^2}{38k^2}$$

$$= \frac{57}{38}$$

$$= \frac{3}{2}$$

Try

Solve the following problems.

1 If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{ab+cd}{ab-cd} = \dfrac{a^2+c^2}{a^2-c^2}$

2 If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2} = \dfrac{xy+yz+zx}{ab+bc+ca}$

3 If $a : b : c = 2 : 3 : 4$, find the value of $\dfrac{ab+bc+ca}{a^2+b^2+c^2}$

Work with a partner

Collaborate with your group to solve question 3, then discuss the answer.

Exercise

Solve the following problems.

1 If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

2 If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{3a+c}{3b+d} = \dfrac{2a-c}{2b-d}$

3 If $\dfrac{x}{a} = \dfrac{y}{b}$, prove that $\dfrac{a^2+x^2}{a^2-x^2} = \dfrac{b^2+y^2}{b^2-y^2}$

4 If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $\dfrac{x+y+z}{a+b+c} = \dfrac{x}{a}$

5 If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $\dfrac{x+2y+3z}{a+2b+3c} = \dfrac{x+y+z}{a+b+c}$

6 If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $(a^2+b^2+c^2)\left(x^2+y^2+z^2\right) = (ax+by+cz)^2$

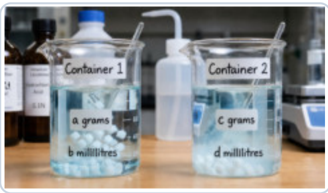
7 If $a : b : c = 3 : 5 : 2$, find the value of $\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}$

8 If $x : y : z = 2 : -3 : 4$, find the value of $\dfrac{x^2+y^2+z^2}{xy+yz+zx}$

9 If $\dfrac{x}{7} = \dfrac{y}{3} \neq 0$, find the value of $\dfrac{x^2+5y^2}{x^2+xy+y^2}$

★ Challenge — Adjusting a Chemical Solution

Imagine you are a chemist with two containers of saltwater solution. Container 1 has a grams of salt in b millilitres of water; Container 2 has c grams in d millilitres. Both have the exact same concentration, so $\dfrac{a}{b} = \dfrac{c}{d}$



Your supervisor comes in with two quality control tests you need to run:
Test 1 (Total Mass): You need to find the ratio of the total mass (salt + water) to the pure water for both containers.
Test 2 (The Difference): You need to find how the total mass compares to the net difference between the water and the salt (water − salt) for both containers.
Prove that if the starting concentrations are equal, the ratio of their combined mass to their separated difference will also remain perfectly equal:

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

1-13

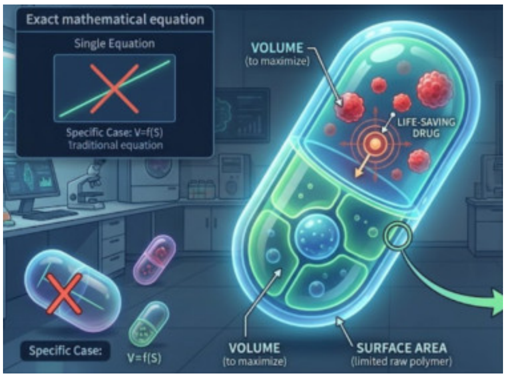
MATHEMATICS

Proving Inequalities (1)

Chapter 1 Algebraic Proofs

Concept 5 · Inequalities

You are a bio-engineer designing a biodegradable capsule to deliver a life-saving drug to a specific tumour inside the human body. To ensure the drug releases at the perfect rate, you need the capsule to have the maximum possible volume, but you have a strict limit on the amount of expensive raw polymer available to create its outer surface area.



How do you find the absolute best design when you can't just use an exact equation to give you a single answer? You use inequalities.

While regular equations tell us exactly when things are equal, inequalities tell us the optimal limits of what is physically possible—the absolute maximums and minimums. In life sciences, economics, and logistics, the most powerful tool for finding these bounds is a beautiful, universal mathematical law known as the AM-GM Inequality.

Guiding question: How can algebraic techniques and the structural properties of positive real numbers be used to establish definitive upper and lower bounds for variable expressions, and how does equality emerge as a special boundary condition?

Learning Outcomes

Prove that algebraic inequalities are correct in various math situations (conditional).

Point!

To prove the inequality LHS > RHS, calculate their difference **LHS − RHS** and show that it is **positive**. (LHS − RHS > 0)

If a condition like $a > b$ is given, rewrite it as $a - b > 0$ to use in your proof.

Warm Up

Prove the following inequalities.

1 If $a > b$ and $x > y$, prove that $2(ax + by) > (a + b)(x + y)$

2 If $x > 3$ and $y > -5$, prove that $xy + 5x > 3y + 15$

Explanation

1 [Proof]

$\text{LHS} - \text{RHS} = 2(ax + by) - (a + b)(x + y)$

$= 2ax + 2by - ax - ay - bx - by$

$= ax - ay - bx + by$

$= a(x - y) - b(x - y)$

$= (a - b)(x - y)$

Since $a > b$ and $x > y$, we have $a - b > 0$ and $x - y > 0$,

$(a - b)(x - y) > 0$

Therefore, $2(ax + by) > (a + b)(x + y)$

> Start with "LHS−RHS = "

> Factor the expression.
> Factor out the common factor from the first and second terms.

> If A > 0 and B > 0, then AB > 0

2 [Proof]

$\text{LHS} - \text{RHS} = xy + 5x - (3y + 15)$

$= xy + 5x - 3y - 15$

$= x(y + 5) - 3(y + 5)$

$= (x - 3)(y + 5)$

Since $x > 3$ and $y > -5$, we have $x - 3 > 0$ and $y + 5 > 0$,

$(x - 3)(y + 5) > 0$

Therefore, $xy + 5x > 3y + 15$

> Start by writing "LHS−RHS = "

> Factor the expression; factor out the common factors from the two preceding and succeeding terms.

Try

Prove the following inequalities.

1 If $a > b$ and $c > d$, prove that $ac + bd > ad + bc$

2 If $x > 2$ and $y > 1$, prove that $xy + 2 > x + 2y$

> **Work with a partner**
> Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following inequalities.

1 If $x > y$, prove that $3x - 4y > 2x - 3y$

2 If $a > c$ and $b > d$, prove that $ab + cd > ad + bc$

3 If $a > 1$ and $b > 1$, prove that $ab + 1 > a + b$

4 If $a > b > x > y$, prove that $ax + by > ay + bx$

★ Challenge — Rearrangement

If $a > b > c > 0$ and $x > y > z > 0$, prove that $ax + by + cz > ay + bz + cx$

1-14 Proving Inequalities (2)

MATHEMATICS

Chapter 1 Algebraic Proofs

Point!

To prove LHS ≥ RHS, calculate their difference and show that

$\text{LHS} - \text{RHS} \geq 0$

To prove an inequality for all real numbers, use **completing the square** to form X^2 or $X^2 + Y^2$

For inequalities with ≥ or ≤, you must state the condition for equality.

For real numbers $X,\ Y$: $\boldsymbol{X}^2 \geq 0$. Equality holds when $\boldsymbol{X} = 0$

$\boldsymbol{X}^2 + \boldsymbol{Y}^2 \geq 0$. Equality holds when $\boldsymbol{X} = 0$ and $\boldsymbol{Y} = 0$

Learning Outcomes

Prove that algebraic inequalities are correct in various math situations (unconditional).

Warm Up

Prove the following inequalities. Also, determine when equality holds.

1 $a^2 + 4b^2 \geq 4ab$

2 $x^2 + y^2 \geq 2(x - y - 1)$

Explanation

1 [Proof]

$\text{LHS} - \text{RHS} = a^2 + 4b^2 - 4ab$

Start with “LHS−RHS”

$= a^2 - 4ab + 4b^2$

Complete the square to form X^2

$= (a - 2b)^2 - 4b^2 + 4b^2$

$= (a - 2b)^2 \geq 0$

$X^2 \geq 0$

Therefore, $a^2 + 4b^2 \geq 4ab$

Also, equality holds when

$a - 2b = 0$, that is, when $a = 2b$

Equality holds when $X = 0$

2 [Proof]

$\text{LHS} - \text{RHS} = x^2 + y^2 - 2(x - y - 1)$

$= x^2 - 2x + y^2 + 2y + 2$

$= (x-1)^2 - 1 + (y+1)^2 - 1 + 2$

$= (x-1)^2 + (y+1)^2 \geq 0$

Therefore, $x^2 + y^2 \geq 2(x - y - 1)$.

Also, equality holds when

$x - 1 = 0$ and $y + 1 = 0$, that is, when $x = 1$ and $y = -1$

Start with "LHS−RHS"

Complete the square for x and y to form $X^2 + Y^2$

$X^2 + Y^2 \geq 0$

Equality holds when $X = 0$ and $Y = 0$

Try

Prove the following inequalities. Also, determine when equality holds.

1 $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$

2 $a^2 + 5b^2 \geq 4ab$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following inequalities. Also, determine when equality holds except for 5.

1 $9x^2 \geq y(6x - y)$

2 $2(x^2 + y^2) \geq (x + y)^2$

3 $(2x - y)^2 \geq 3(x^2 - y^2)$

4 $(2a + 3b)^2 \geq 12ab$

5 $x^2 + 2 > 2x$

6 $x^2 + y^2 - 6x + 8y + 25 \geq 0$

★ Challenge — Never Negative

You are a data analyst at a factory that produces precision drone components using two machines, X and Y. Due to mechanical wear, calibration drifts, and energy spikes, the daily overhead cost (in dollars) based on the settings x and y is modelled by

$$\text{Overhead Cost} = x^2 + 2y^2 - 10x + 12y + 43$$

Prove that no matter what settings x and y are chosen, the factory will never have a negative overhead cost.

1-15
MATHEMATICS

Proving Inequalities (3)

Chapter 1 Algebraic Proofs

Point!

To prove inequalities with radicals, calculate the difference of their squares, $(\text{LHS})^2 - (\text{RHS})^2$

Use the following properties in the proof:

- When $\text{LHS} > 0, \text{RHS} > 0$: $(\text{LHS})^2 > (\text{RHS})^2 \iff \text{LHS} > \text{RHS}$

$(\text{LHS})^2 \geq (\text{RHS})^2 \iff \text{LHS} \geq \text{RHS}$

- When $a > 0$, $\sqrt{a} > 0$

Learning Outcomes

Prove that algebraic inequalities are correct in various math situations (including those with roots).

Warm Up

Prove the following inequalities. Also, determine when equality holds for 1.

1 If $a > 0$ and $b > 0$, $\sqrt{\dfrac{a^2+b^2}{2}} \geq \dfrac{a+b}{2}$

2 If $a > 0$ and $b > 0$, $\sqrt{a} + 4\sqrt{b} > \sqrt{a+16b}$

Explanation

1 [Proof]

$$(\text{LHS})^2 - (\text{RHS})^2 = \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2 - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2}{2} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{a^2-2ab+b^2}{4}$$

$$= \left(\frac{a-b}{2}\right)^2 \geq 0$$

Thus, $\left(\sqrt{\dfrac{a^2+b^2}{2}}\right)^2 \geq \left(\dfrac{a+b}{2}\right)^2$ ...... (i)

Since a > 0, b > 0, $\sqrt{\dfrac{a^2+b^2}{2}} > 0$, $\dfrac{a+b}{2} > 0$ ...... (ii)

From (i) and (ii), $\sqrt{\dfrac{a^2+b^2}{2}} \geq \dfrac{a+b}{2}$

Also, equality holds when

$\dfrac{a-b}{2} = 0$, that is, when $a = b$.

Calculate $(\text{LHS})^2 - (\text{RHS})^2$

$X^2 \geq 0$

Showed $(\text{LHS})^2 \geq (\text{RHS})^2$

Must show LHS > 0, RHS > 0

Proved LHS $\geq$ RHS

Equality holds when $X = 0$

2 [Proof]

$(\text{LHS})^2 - (\text{RHS})^2 = \left(\sqrt{a} + 4\sqrt{b}\right)^2 - \left(\sqrt{a+16b}\right)^2$

$= a + 8\sqrt{ab} + 16b - (a + 16b)$

$= a + 8\sqrt{ab} + 16b - a - 16b$

$= 8\sqrt{ab} > 0$

Thus, $(\sqrt{a} + 4\sqrt{b})^2 > \left(\sqrt{a+16b}\right)^2$ ......(i)

Since $a > 0,\ b > 0,\ \sqrt{a} + 4\sqrt{b} > 0,\ \sqrt{a+16b} > 0$ ...... (ii)

From (i) and (ii), $\sqrt{a} + 4\sqrt{b} > \sqrt{a+16b}$

Calculate $(\text{LHS})^2 - (\text{RHS})^2$

$\sqrt{ab} > 0$

Must show $\text{LHS} > 0,\ \text{RHS} > 0$

Try

Prove the following inequalities. Also, determine if equality holds for 1

1 If $a > 0$ and $b > 0$, $\sqrt{2(a+b)} \geq \sqrt{a} + \sqrt{b}$

2 If $a > 0$ and $b > 0$, $2\sqrt{a} + \sqrt{b} > \sqrt{4a+b}$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following inequalities. Also, determine if equality holds for 1 to 3

1 If $a > 0,\ 2 + a \geq \sqrt{3+6a}$

2 If $a > 0$ and $b > 0$, $\sqrt{2(a^2+b^2)} \geq a + b$

3 If $a > 0$ and $b > 0$, $\sqrt{\dfrac{a+b}{2}} \geq \dfrac{\sqrt{a} + \sqrt{b}}{2}$

4 If $a > 0$ and $b > 0$, $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$

5 If $a > 0$ and $b > 0$, $2\sqrt{a} + 3\sqrt{b} > \sqrt{4a+9b}$

6 If $a > b > 0$, $\sqrt{a-b} > \sqrt{a} - \sqrt{b}$

★ Challenge — A Root Inequality

If $a > b$, where a and b are two positive numbers, prove that $\sqrt{a-b} < \sqrt{a} - \dfrac{b}{2\sqrt{a}}$

1-16

MATHEMATICS

Proving Inequalities (4) (AM-GM Inequality)

Chapter 1 Algebraic Proofs

Point!

The Arithmetic-Geometric Mean (AM-GM) Inequality

When $\boldsymbol{a > 0}$, $\boldsymbol{b > 0}$, $\frac{a+b}{2} \geq \sqrt{ab}$, or $\boldsymbol{a + b \geq 2\sqrt{ab}}$

Equality holds when $\boldsymbol{a = b}$

The AM-GM inequality is highly useful when the product of the terms simplifies to a constant.

Example: $a + \frac{1}{a} \Rightarrow a \times \frac{1}{a} = 1$

Learning Outcomes

Applies the arithmetic-geometric mean inequality to prove inequalities and find max values.

Warm Up

Prove the following inequalities. Also, determine if equality holds.

1 If $a > 0$ and $b > 0$, $\frac{b}{a} + \frac{a}{b} \geq 2$

2 If $a > 0$ and $b > 0$, $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$

Explanation

1 [Proof]

Since $\frac{b}{a} > 0$, $\frac{a}{b} > 0$,

Always write this

By the AM-GM inequality,

$\frac{b}{a} + \frac{a}{b} \geq 2\sqrt{\frac{b}{a} \times \frac{a}{b}}$

$\frac{b}{a} + \frac{a}{b} \geq 2$

Therefore, $\frac{b}{a} + \frac{a}{b} \geq 2$

Also, equality holds

when $\frac{b}{a} = \frac{a}{b}$

Multiply both sides by ab

$b^2 = a^2$

Since $a > 0$, $b > 0$, $a = b$

Thus, equality holds when $a = b$

2 [Proof]

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=a\times\frac{1}{a}+\frac{a}{b}+\frac{b}{a}+b\times\frac{1}{b}$$

$$=\frac{a}{b}+\frac{b}{a}+2\ldots\ldots\text{(i)}$$

Since $\frac{a}{b}>0,\ \frac{b}{a}>0$

Always write this

By the AM-GM inequality,

$$\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\times\frac{b}{a}}$$

$$\frac{a}{b}+\frac{b}{a}\geq 2$$

Add 2 to both sides to match (i)

$$\frac{a}{b}+\frac{b}{a}+2\geq 4$$

Showed (Expression (i)) ≥ 4

Therefore, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$

Also, equality holds

when $\frac{a}{b}=\frac{b}{a}$

Multiply both sides by ab

$a^2=b^2$

Since $a>0,\ b>0,\ a=b$

Thus, equality holds when $a=b$

Try

Prove the following inequalities. Also, determine if equality holds.

1 If $a>0$ and $b>0$, $\frac{b}{3a}+\frac{12a}{b}\geq 4$

2 If $a>0$ and $b>0$, $\left(a+\frac{1}{b}\right)\left(b+\frac{9}{a}\right)\geq 16$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Prove the following inequalities. Also, determine if equality holds.

1 If $a>0$, $a+\frac{4}{a}\geq 4$

2 If $a>0$ and $b>0$, $3ab+\frac{4}{ab}\geq 4\sqrt{3}$

3 If $a>0$ and $b>0$, $a+b+\frac{9}{a+b}\geq 6$

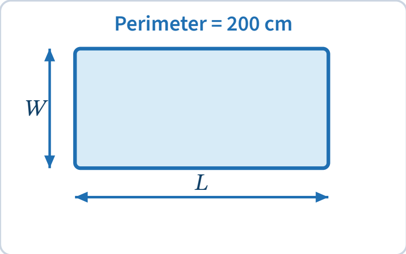
4 If $x>0$, $\left(x+\frac{1}{x}\right)\left(x+\frac{4}{x}\right)\geq 9$

5 If $a>0$ and $b>0$, $(a+8b)\left(\frac{2}{a}+\frac{1}{b}\right)\geq 18$

6 If $a>0$ and $b>0$, $\left(a+\frac{3}{b}\right)\left(b+\frac{3}{a}\right)\geq 12$

★ Challenge — Maximum Area

A rectangle has a perimeter of 200 cm. Using the relation between A.M. and G.M., prove that the area of the rectangle ≤ 2500 cm^2.



Reflect

We proved in the lesson that the AM-GM Inequality ($\frac{a+b}{2} \geq \sqrt{ab}$) always holds true, provided that the numbers are non-negative ($a,\ b \geq 0$).

Question for Reflection: What happens if we try to break this condition and apply the inequality to negative numbers? Suppose that $a = -2$ and $b = -8$.

1. Calculate their Arithmetic Mean (AM).
2. Attempt to calculate their Geometric Mean (GM).
3. Does the inequality sign (≥) still hold true and remain valid?

Explain the algebraic and logical breakdown that occurs here, and why defining "domains and restrictions" in inequalities is vital to prevent medical and engineering models from collapsing in the real world.

2-1
MATHEMATICS

Sequences and General Terms

Chapter 2 Sequences

Concept 1 · Arithmetic sequences

Imagine you are a personal trainer designing a recovery program for an injured athlete. You have them jog for 10 minutes a day on week one, increasing it by exactly 3 minutes every subsequent week (13 minutes, then 16, then 19...).

This constant linear increase is an arithmetic sequence in action. In medicine and biology, these fixed patterns govern critical processes like the hourly rate at which the liver clears toxins or how cells divide over time. Instead of adding the schedule week by week to find your patient's training duration on Week 24 or their total accumulated running time over several months, arithmetic formulas deliver these exact answers instantly.

Guiding Question: How can constant rates of additive change be modeled mathematically to determine both specific individual terms and total cumulative sums across a sequence?

Learning Outcomes

Find the general term for a sequence, and deduce some of its terms.

Point!

! A list of numbers arranged in a row is called a sequence, and each number in the sequence is called **a term** .

The terms of a sequence are called the first term , the 2nd term, the 3rd term, etc., in order from the beginning. The term in the n-th position is called the n—th term.

! To express a sequence generally, we write it as follows by attaching the term number to a single letter.

$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$

a_1 is the first term, a_n is the n—th term.

This sequence is also denoted as $\{a_n\}$.

Example: $\{a_n\}: 2,\ 4,\ 8,\ 16,\ \dots$

$a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16,\ \dots$

! When a_n is expressed as an expression in terms of n, substituting 1, 2, 3, … for n gives $a_1,\ a_2,\ a_3,\ \dots$

Such an expression in terms of n is called **the general term** of the sequence $\{a_n\}$.

Example: $a_n = -2n+7$

$-2n+7$ is a general term.

$a_1 = -2\times 1 + 7 = 5 \qquad a_2 = -2\times 2 + 7 = 3 \qquad a_3 = -2\times 3+7 = 1$

Warm Up

Solve the following problems.

1 When the general term of the sequence $\{a_n\}$ is given by $a_n = \frac{n}{n+1}$, find the first term through the 3rd term of this sequence.

2 Deduce the general term of the following sequences $\{a_n\}$.

(a) $0,\ 1,\ 4,\ 9,\ 16,\ \ldots$ **(b)** $2,\ -4,\ 6,\ -8,\ 10,\ \ldots$ **(c)** $1,\ \frac{2}{3},\ \frac{3}{5},\ \frac{4}{7},\ \frac{5}{9},\ \ldots$

Explanation

1 Substitute $n = 1,\ 2,\ 3$ into $a_n = \frac{n}{n+1}$

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad a_3 = \frac{3}{3+1} = \frac{3}{4}$$

2 Create a formula by considering the regularity.

(a) Since it is (term number − 1)^2, $a_n = (n-1)^2$

(b) For sequences with alternating signs, consider the regularity of the sign part and the number part separately.

Regarding the sign, since the first term is " + " and it alternates, $(-1)^{n-1}$

Regarding the number, since it is 2 × (term number), $2n$

Therefore, $a_n = (-1)^{n-1} \times 2n$

(c) For fractional sequences, consider the regularity of the denominator and numerator separately.

The denominator is 2 × (term number) − 1, so $2n-1$

The numerator is (term number), so n

Therefore, $a_n = \frac{n}{2n-1}$

If the first term is " − " and it alternates, it is $(-1)^n$.

Even numbers are $2n$.

Odd numbers are $2n-1$

Try

Solve the following problems.

1 When the general term of the sequence $\{a_n\}$ is given by the following formulas, find the first term through the 3rd term of the sequence.

(a) $a_n = 2n-3$ **(b)** $a_n = 2 \times 3^n$ **(c)** $a_n = n^2 - 7n$ **(d)** $a_n = \frac{n}{2n+1}$

2 Deduce the general term of the following sequences $\{a_n\}$.

(a) $0,\ 1,\ 8,\ 27,\ 64,\ \ldots$ **(b)** $-3,\ 6,\ -9,\ 12,\ -15,\ \ldots$

(c) $2,\ 5,\ 8,\ 11,\ 14,\ \ldots$ **(d)** $1,\ \frac{3}{4},\ \frac{5}{9},\ \frac{7}{16},\ \frac{9}{25},\ \ldots$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 When the general term of the sequence $\{a_n\}$ is given by the following formulas, find the first term through the 3rd term of the sequence.

(a) $a_n = 5n-8$ **(b)** $a_n = -3n+1$ **(c)** $a_n = -2n-4$ **(d)** $a_n = 3^n-1$

(e) $a_n = (-3)^n$ **(f)** $a_n = 3\times(-2)^n$ **(g)** $a_n = n^2-1$ **(h)** $a_n = n^2+n+1$

(i) $a_n = n^2+2n+4$ **(j)** $a_n = \dfrac{1}{3n-2}$ **(k)** $a_n = \dfrac{2n}{n+3}$ **(l)** $a_n = \dfrac{n+1}{2n-1}$

2 Deduce the general term of the following sequences $\{a_n\}$.

(a) $0,\ 2,\ 4,\ 6,\ 8,\ \dots$ **(b)** $1,\ 4,\ 9,\ 16,\ 25,\ \dots$

(c) $0,\ 3,\ 6,\ 9,\ 12,\ \dots$ **(d)** $-5,\ 10,\ -15,\ 20,\ -25,\ \dots$

(e) $1,\ -\dfrac{1}{3},\ \dfrac{1}{5},\ -\dfrac{1}{7},\ \dfrac{1}{9},\ \dots$ **(f)** $-1,\ 4,\ -7,\ 10,\ -13,\ \dots$

(g) $7^2,\ 8^2,\ 9^2,\ 10^2,\ 11^2,\ \dots$ **(h)** $4,\ 9,\ 14,\ 19,\ 24,\ \dots$

(i) $6,\ 13,\ 20,\ 27,\ 34,\ \dots$ **(j)** $\dfrac{1}{2},\ \dfrac{2}{3},\ \dfrac{3}{4},\ \dfrac{4}{5},\ \dfrac{5}{6},\ \dots$

(k) $1\times3,\ 2\times9,\ 3\times27,\ 4\times81,\ 5\times243,\ \dots$

★ Challenge

$$1\times\frac{1}{2},\ 2\times\frac{1}{3},\ 3\times\frac{1}{4},\ 4\times\frac{1}{5},\ 5\times\frac{1}{6},\ \dots$$

Deduce the general term then find the 20th term.

2-2
MATHEMATICS

Arithmetic Sequence (1)

Chapter 2 Sequences

Point!

❗ A sequence obtained by successively adding a constant number d to each preceding term is called **an arithmetic sequence**, and the constant number d is called **the common difference**.

Example: An arithmetic sequence with the first term 1 and the common difference 3

$1, \xrightarrow{+3} 4, \xrightarrow{+3} 7, \xrightarrow{+3} 10, \xrightarrow{+3} 13, \ldots$

❗ The general term of an arithmetic sequence with the first term a and the common difference d is $a_n = \boldsymbol{a + (n-1)d}$

$$\underbrace{a \xrightarrow{+d} a_2 \xrightarrow{+d} a_3, \ldots, a_{n-1} \xrightarrow{+d} a_n}$$

a_n is obtained by adding "d" to "a" exactly (n – 1) times

Learning Outcomes

Calculate the general term of an arithmetic sequence, and find some specific terms in it.

Warm Up

Solve the following problems.

1 Find the first term and the common difference of the following arithmetic sequences.

(a) $2, \; 5, \; 8, \; 11, \; \ldots$ (b) $3, \; -2, \; -7, \; -12, \; -17, \; \ldots$

2 The following sequences are arithmetic sequences. Find the common difference and the numbers that fit in A and B.

(a) $A, \; 2, \; -1, B, \; \ldots$ (b) $13, A, \; B, \; -2, \; \ldots$

3 For the arithmetic sequence $\{a_n\}$ with the first term 8 and the common difference −6, find the general term. Also, find the 20th term.

Explanation

1 (a) First term: 2, Common difference: 3

$2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11, \ldots$

(b) First term: 3, Common difference: −5

$3 \xrightarrow{-5} -2 \xrightarrow{-5} -7 \xrightarrow{-5} -12 \xrightarrow{-5} -17, \ldots$

2 (a) $A, \xrightarrow{-3} 2, \xrightarrow{-3} -1, \xrightarrow{-3} B, \xrightarrow{-3} \ldots$

Common difference: -3, **A**: 5, **B**: -4

(b) $\underbrace{13, \xrightarrow{d} A, \xrightarrow{d} B, \xrightarrow{d} -2,}_{-15} \xrightarrow{d} \ldots$

Let the expected common difference be d, $3d = -15$ Therefore, $d = -5$

Common difference: −5, **A**: 8, **B**: 3

3 The general term is found by substituting $a = 8$, $d = -6$ into

$a_n = a + (n-1)\,d.$

$a_n = 8 + (n-1) \times (-6)$

$= 8 - 6n + 6$

$= -6n + 14$ $\underline{a_n = -6n+14}$

The 20th term is found by substituting $n = 20$ into

$a_n = -6n + 14$

$a_{20} = -6 \times 20 + 14$

$= -106$ $\underline{a_{20} = -106}$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Solve the following problems.

1 Find the first term and the common difference of the following arithmetic sequences.

(a) $1, 5, 9, 13, 17, \ldots$ (b) $23, 19, 15, 11, 7, \ldots$

2 The following sequences are arithmetic sequences. Find the common difference and the numbers that fit in A and B.

(a) $2, 9, \mathrm{A}, \mathrm{B}, \ldots$ (b) $\mathrm{A}, 17, \mathrm{B}, 5, \ldots$

3 For the following arithmetic sequences $\{a_n\}$, find the general term. Also, find the 7$^{th.}$ term.

(a) First term 4, common difference 3 (b) First term 2, common difference −3

Exercise

Solve the following problems.

1 Find the first term and the common difference of the following arithmetic sequences.

(a) $3, 5, 7, 9, 11, \ldots$ (b) $-13, -9, -5, -1, 3, \ldots$

(c) $-2, 1, 4, 7, 10, \ldots$ (d) $10, 7, 4, 1, -2, \ldots$

(e) $6, \frac{16}{3}, \frac{14}{3}, 4, \frac{10}{3}, \ldots$ (f) $-4, -\frac{9}{2}, -5, -\frac{11}{2}, -6, \ldots$

2 The following sequences are arithmetic sequences. Find the common difference and the numbers that fit in A and B.

(a) 8, 15, A, B, … (b) 7, 13, A, B, …

(c) A, 9, 4, B, … (d) A, 15, B, 9, …

(e) 11, A, B, 20, … (f) 17, A, B, 5, C, …

3 For the following arithmetic sequences $\{a_n\}$, find the general term. Also, find the 7th term.

(a) The first term 5, and the common difference 4

(b) The first term − 2, and the common difference 3

(c) The first term 10, and the common difference −3

(d) The first term $-\frac{1}{2}$, and the common difference $-\frac{1}{2}$

4 For the following arithmetic sequences $\{a_n\}$, find the general term. Also, find the 10th term.

(a) The first term −7, and the common difference 3

(b) The first term −2, and the common difference −3

(c) The first term 7, and the common difference $\frac{1}{2}$

(d) The first term $\frac{2}{3}$, and the common difference $-\frac{1}{3}$

★ Challenge

In an arithmetic sequence, the sum of the 3rd. and 7th. terms is 38, and the ratio of the 4th. term to the 12th. term is $\frac{2}{5}$ Find the first term of this sequence and the common difference.

2-3

MATHEMATICS

Arithmetic Sequence (2)

Chapter 2 Sequences

Point!

! The general term a_n of an arithmetic sequence $\{a_n\}$ is $\boldsymbol{a + (n-1)d}$

! When finding the general term of an arithmetic sequence, first find the first term a and the common difference d

! When the sequence $a,\ b,\ c$ is an arithmetic sequence in this order:

$2b = a + c$

Learning Outcomes

Calculate the general term of an arithmetic sequence, and find some specific terms in it.

Considering $a, \overset{+d}{\curvearrowright} b, \overset{+d}{\curvearrowright} c$:

$2b = (a + d) + (c - d)$

$\quad = a + c$

Warm Up

Solve the following problems.

1 Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd term is 10 and the 8th term is 50

2 For the arithmetic sequence $\{a_n\}$ with the common difference 4 and the 8th term 33, solve the following problems.

(a) Find which term number is 113

(b) Find the first term number that is greater than 500

3 When the 3 numbers $x, 6, x^2$ form an arithmetic sequence in this order, find the value of x

Explanation

1 Let the general term be $a_n = a + (n-1)d$.

Since $a_3 = 10$, $10 = a + (3-1)d$. Simplifying this, $10 = a + 2d$. ......(i)

Since $a_8 = 50$, $50 = a + (8-1)d$. Simplifying this, $50 = a + 7d$. ......(ii)

Solving (i) and (ii) as a system of equations, $a = -6,\ d = 8$

Therefore, the general term is $a_n = -6 + (n-1) \times 8$

$= 8n - 14$ $\qquad$ $\boldsymbol{a_n = 8n - 14}$

2 Let the general term of $\{a_n\}$ be $a_n = a + (n-1)d$.

First find the general term

Since the common difference is 4 and $a_8 = 33$, $33 = a + (8-1) \times 4$

Solving this, $a = 5$

Therefore, the general term of $\{a_n\}$ is $a_n = 5 + (n-1) \times 4$

$a_n = 4n + 1$

(a) We need to find n such that $a_n = 113$ $113 = 4n+1$ Solving this, $n = 28$

Therefore, **the 28th term**.

(b) We need to find the smallest natural number n such that $a_n > 500$

$4n+1 > 500$ Solving this, $n > \frac{499}{4} = 124.75$

Convert to decimal to consider

Therefore, **the 125th term**.

3 Since $x,\ 6,\ x^2$ form an arithmetic sequence in this order:

$2 \times 6 = x + x^2$

$2b = a + c$

Solving this, $x = -4, 3$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Solve the following problems.

1 Find the general term of the following arithmetic sequences $\{a_n\}$.

(a) The first term is −5, the 17th term is 75

(b) The common difference is 3, the 16th term is 81

(c) The 6th term is 2, the 15th term is −25

2 For the arithmetic sequence $\{a_n\}$ with first term 1 and common difference 3, solve the following problems.

(a) Determine which term is 40

(b) Find the first term that is greater than 100

3 When the 3 numbers $x^2, x, -8$ form an arithmetic sequence in this order, find the value of x

Exercise

Solve the following problems.

1 Find the general term of the following arithmetic sequences $\{a_n\}$.

(a) The first term is 6, the 4th term is 18

(b) The first term is 20, the 8th term is −1

(c) The first term is −8, the 8th term is 55

(d) The first term is −18, the 10th term is −63

(e) The common difference is 2, the 8th term is 4

(f) The common difference is −4, the 5th term is −7

(g) The common difference is −7, the 6th term is −24

(h) The common difference is $\frac{2}{5}$, the 30th term is $\frac{23}{5}$.

(i) The 4th term is 14, the 10th term is 62

(j) The 3rd term is 30, the 10th term is 2

(k) The 10th term is 49, the 23rd term is 10

(l) The 3rd term is 6, the 10th term is 62

2 For the arithmetic sequence $\{a_n\}$ with the common difference −5 and the 13th term −7, solve the following problems.

(a) Determine which term is −52

(b) Find the first term that is smaller than −300

3 For the arithmetic sequence $\{a_n\}$ where the 10th term is −20 and the 20th term is −50, solve the following problems.

(a) Determine which term is −110

(b) Find the first term that is smaller than −500

4 When the 3 numbers a, 6, $2a$ form an arithmetic sequence in this order, find the value of a

5 When the 3 numbers $-3, x, x^2$ form an arithmetic sequence in this order, find the value of x

★ Challenge

When the 4 numbers $2, x, y, z$ form an arithmetic sequence in this order, and $2x,\ y+1,\ z-y$ also form an arithmetic sequence in this order, find the values of $x,\ y,\ z$

2-4

MATHEMATICS

Sum of Arithmetic Sequence (1)

Chapter 2 Sequences

Point!

! In a sequence with n terms, n is called **the number of terms** , and the last term is called **the last term** .

Let S_n be the sum from the first term to the n–th term of an arithmetic sequence.

When the first term is a, the last term is l, the number of terms is n,

$$S_n = \frac{1}{2}n(a+l)$$

When the first term is a, the common difference is d, the number of terms is n,

$$S_n = \frac{1}{2}n\{2a+(n-1)d\}$$

! When the number of terms n is unknown, first find n using

$l = a + (n-1)d$

Learning Outcomes

Calculate the general term of an arithmetic sequence.
Find some specific terms in an arithmetic sequence. Find the sum of an arithmetic sequence.

Substitute $l = a + (n-1)d$ into the above formula.

Warm Up

Solve the following problems.

1 Find the sum of the following arithmetic sequences.

(a) The sum S_7 of a sequence with the first term 8, the last term 22, the number of terms 7

(b) The sum S of a sequence with the first term 2, the common difference 3, the last term 47

(c) The sum S_{13} of a sequence with the first term 20, the common difference −5, the number of terms 13

(d) The sum S_n from the first term to the n-th term of a sequence with the first term 1, the common difference 4

2 For the arithmetic sequence 90, 84, 78, ..., find the sum from the 10th term to the 20th term.

Explanation

1 (a) Since $a = 8, l = 22, n = 7$:

$S_7 = \frac{1}{2} \times 7(8+22)$

$= 105$

(b) First find the number of terms. Let n be the number of terms.

Since the first term is 2, the common difference is 3, the last term is 47:

$47 = 2+(n-1) \times 3$

$l = a + (n-1)d$

Solving this, $n = 16$

$S = \frac{1}{2} \times 16(2+47)$

$= 392$

(c) Since $a= 20, d= -5, n = 13$:

$S_{13} = \frac{1}{2} \times 13\{2 \times 20 + (13-1) \times (-5)\}$

$= -130$

(d) Since $a= 1, d = 4$:

$S_n = \frac{1}{2}n\{2 \times 1+(n-1) \times 4\}$

Since the number of terms is n, evaluate the expression in terms of n.

$= \frac{1}{2}n(2+4n-4) \qquad = \frac{1}{2}n(4n-2)$

$= 2n^2 - n$

2 The required sum can be calculated by $S_{20} - S_9$

$$\underbrace{\underbrace{a_1 + a_2 + a_3 + \dots + a_9}_{S_9} + \underbrace{a_{10} + \dots + a_{20}}_{\text{Required Sum}}}_{S_{20}}$$

Since $a= 90, d= -6$:

$S_{20} = \frac{1}{2} \times 20\{2 \times 90 + (20-1) \times (-6)\} = 660$

$S_9 = \frac{1}{2} \times 9\{2 \times 90 + (9-1) \times (-6)\} = 594$

Therefore, $660 - 594 = \underline{66}$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Solve the following problems.

1 Find the sum of the following arithmetic sequences.

(a) The sum S_{10} of a sequence with the first term 7, the last term 13, the number of terms 10

(b) The sum S of a sequence with the first term 5, the common difference 3, the last term 44

(c) The sum S_{12} of a sequence with the first term − 6, the common difference 3, the number of terms 12

(d) The sum S_n from the first term to the n-th term of a sequence with the first term 2, the common difference −3

2 For the arithmetic sequence with first term 48 and common difference −2, find the sum from the 20th term to the 30th term

Exercise

Solve the following problems.

1 Find the sum of the following arithmetic sequences.

(a) The sum S_{10} of sequence: the first term 3, the last term 21, the number of terms 10

(b) The sum S_8 of sequence: the first term 5, the last term 33, the number of terms 8

(c) The sum S_{10} of sequence: the first term 2, the last term 29, the number of terms 10

(d) The sum S of sequence: the first term 3, the common difference 4, last term 75

(e) The sum S of sequence: the first term 2, the common difference 5, last term 57

(f) The sum S of sequence: the first term 4, the common difference 3, last term 100

(g) The sum S_{15}: the first term 10, the common difference −3, the number of terms 15

(h) The sum S_{11}: the first term −4, the common difference 4, the number of terms 11

(i) The sum S_{18}: the first term 3, the common difference 2, the number of terms 18

(j) The sum S_n from the first term to the n-th term: the first term 0, the common difference 5

(k) The sum S_n from the first term to the n-th term: the first term 3, the common difference −4

(l) The sum S_n from the first term to the n-th term: the first term −4, the common difference −2

2 For the arithmetic sequence −9, −3, 3, ..., find the sum from the 11th term to the 20th term.

3 For the arithmetic sequence with the first term −53 and the common difference 4, find the sum from the 10th term to the 25th term.

4 For the arithmetic sequence 74, 69, 64, ..., find the sum from the 8th term to the 18th term.

★ Challenge

For the arithmetic sequence 94, 89, 84, ..., find the sum of the odd terms from the first term to the 11th term.

2-5

MATHEMATICS

Sum of Arithmetic Sequence (2) (Sum of Natural Numbers)

Chapter 2 Sequences

Point!

! Use arithmetic sequences for problems involving sums of natural numbers or multiples.

! The sum S_n of an arithmetic sequence from the first term to the n—th term is: When the first term is a, the last term is l, and the number of terms is n: $S_n = \frac{1}{2}n(a+l)$

When the first term is a, the common difference is d, and the number of terms is n: $S_n = \frac{1}{2}n\{2a+(n-1)d\}$

! When the number of terms n is unknown, first find n using $l = a + (n-1)d$

Learning Outcomes

Calculate the general term of an arithmetic sequence.
Find some specific terms in an arithmetic sequence.
Find the sum of an arithmetic sequence (natural numbers).

Warm Up

Find the following sums.

1 The sum of natural numbers from 1 to 100

2 The sum of natural numbers from 101 to 200

3 The sum of multiples of 3 among 2-digit natural numbers.

Explanation

1 Since the required sum is 1 + 2 + 3 + … + 100:

Calculate the sum of an arithmetic sequence with the first term 1, the last term 100, and the number of terms 100

$1+2+3+\ldots+100 = \frac{1}{2} \times 100(1+100)$

$= \underline{5050}$

2 Since the required sum is 101 + 102 + 103 + … + 200:

Calculate the sum of an arithmetic sequence with the first term 101, the last term 200, and the number of terms 100

$101+102+103+\ldots+200 = \frac{1}{2} \times 100(101+200)$

$= \underline{15050}$

Write out the terms concretely and consider the first term, the last term, and the number of terms.

$S_n = \frac{1}{2}n(a+l)$

There are 100 natural numbers from 101 to 200

3 Since the required sum is $12 + 15 + 18 + \ldots + 99$:

Calculate the sum of an arithmetic sequence with the first term 12, the common difference 3, and the last term 99

Since the number of terms n is unknown, find n using $l = a + (n-1)d$.

First, find the number of terms. Let n be the number of terms.

$99 = 12 + (n - 1) \times 3$. Solving this, $n = 30$

$$12 + 15 + 18 + \ldots + 99 = \frac{1}{2} \times 30(12 + 99)$$

$= \underline{1665}$

Try

Work with a partner
Collaborate with your group to solve the try section then discuss the answer.

Find the following sums.

1 The sum of natural numbers from 1 to 200

2 The sum of natural numbers from 201 to 300

3 The sum of all 2-digit natural numbers that are multiples of 5

4 The sum of all 2-digit natural numbers that leave a remainder of 3 when divided by 5

Exercise

Find the following sums.

1 The sum of natural numbers from 1 to 150

2 The sum of natural numbers from 81 to 150

3 The sum of natural numbers from 301 to 400

4 The sum of 2-digit natural numbers.

5 The sum of 2-digit odd numbers.

6 The sum of multiples of 4 from 1 to 100

7 The sum of all natural numbers from 1 to 100 that leave a remainder of 1 when divided by 4

8 The sum of all multiples of 6 from 10 to 100

9 The sum of all natural numbers from 10 to 100 that are not divisible by 6

★ Challenge

Find The sum of numbers which are not divisible by 4 from 1 to 100

2-6

MATHEMATICS

Sum of Arithmetic Sequence (3)

Chapter 2 Sequences

Learning Outcomes

Find the sum of an arithmetic sequence.
Determine the number of terms in an arithmetic sequence.

Point!

! The sum S_n of an arithmetic sequence from the first term to the n–th term is:

When the first term is a, the last term is l, and the number of terms is n :

$S_n = \frac{1}{2}n(a + l)$

When the first term is a, the common difference is d, and the number of terms is n : $S_n = \frac{1}{2}n\{2a + (n-1)d\}$

! To find the maximum value of the sum of an arithmetic sequence with a negative common difference, calculate **the sum of the positive terms** .

Example: Arithmetic sequence with first term 11, common difference −3

$11,\ 8,\ 5,\ 2,\ -1,\ -4,\ \dots$

The sum up to here is maximum.

Warm Up

Solve the following problems.

1 In an arithmetic sequence with the first term 4 and the common difference 3, find how many terms from the first term sum up to 50

2 Find the general term of the arithmetic sequence $\{a_n\}$ where the first term is −60 and the sum from the first term to the 15th term is −60

3 For an arithmetic sequence with the first term 65 and the common difference −4, find the number of terms from the start that gives the maximum sum. Also, find the corresponding sum.

Explanation

1 Assume the sum from the first term to the n-th term of an arithmetic sequence with the first term 4 and the common difference 3 is 50

$50 = \frac{1}{2}n\{2 \times 4 + (n-1) \times 3\}$

Simplifying gives $3n^2 + 5n - 100 = 0$

Solving this gives $n = 5,\ -\frac{20}{3}$. Since n is a natural number, **the 5th term**.

2 Let d be the common difference of the required arithmetic sequence.

Since the first term is − 60 and the sum to the 15th term is − 60:

$-60 = \frac{1}{2} \times 15\,\{2 \times (-60) + (15-1)d\}$

Solving this gives $d = 8$

Therefore, the general term is $a_n = -60 + (n-1) \times 8$

Simplifying gives $\boldsymbol{a_n = 8n - 68}$

3 The general term of this sequence is $a_n = 65 + (n-1) \times (-4)$

$= -4n + 69$

Here, solving $-4n + 69 > 0$ gives $n < \frac{69}{4} = 17.25$

In other words, terms from the first to the 17th are positive, and terms from the 18th onwards are negative.

Therefore, the sum up to the 17th term is maximum. **the 17th term**

Also, the sum at that time is the sum of an arithmetic sequence with the first term 65, the common difference −4, and the number of terms 17:

$\frac{1}{2} \times 17\{2 \times 65 + (17-1) \times (-4)\} = \mathbf{561}$

Common Mistakes

$S_n = \frac{1}{2}n\{2 \times 65 + (n-1)(-4)\} > 0$

You are solving for when the sum is positive ($S_n > 0$) instead of finding until which term the sequence remains positive ($a_n > 0$).

To find the general term, you need the first term a and the common difference d.

Find up to which term the terms are positive.

$\underbrace{a_1,\ a_2,\ a_3,\ \dots,\ a_{17},}_{\text{positive terms}}\quad \underbrace{a_{18},\ a_{19},\ \dots}_{\text{negative terms}}$

Try

Solve the following problems.

1 In an arithmetic sequence with the first term 17 and the common difference −2, find how many terms from the first term sum up to −63

2 Find the general term of the arithmetic sequence $\{a_n\}$ where the first term is 8 and the sum from the first term to the 13th term is −130

3 For an arithmetic sequence with the first term 70 and the common difference −4, find the number of terms from the start that gives the maximum sum. Also, find the corresponding sum.

4 Find the general term of the arithmetic sequence $\{a_n\}$ where the sum from the first term to the 10th term is 10, and the sum from the first term to the 20th term is 40

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find n such that the sum of the first n terms of an arithmetic sequence with the first term 3 and the common difference 2 is 63

2 Find n such that the sum of the first n terms of an arithmetic sequence with the first term 24 and the common difference −4 is −28

3 Find the general term of the arithmetic sequence $\{a_n\}$ with the first term 4 and the sum of the first 6 terms 114

4 For a sequence with the first term 20, the common difference −6, find the value of n that gives the maximum sum, and the corresponding sum.

5 For a sequence with the first term 126, the common difference −13, find the value of n that gives the maximum sum, and the corresponding sum.

6 Find the general term of sequence $\{a_n\}$ where the 3rd term 10 and the sum of first 10 terms 175

7 Find the general term of $\{a_n\}$ where sum of first 6 terms is 3 and the sum of first 10 terms is 65

8 Find the general term of $\{a_n\}$ where sum of first 20 terms is 400 and the sum of first 30 terms is 900

★ Challenge

The 15^{th} term of the arithmetic sequence is 33 and the 50^{th} term is 103. What is the first 79 terms of the arithmetic sequence?

Reflect

"If the water level in a lake decreases by a fixed amount each day, how can understanding arithmetic sequences help us predict when the water will run out?"

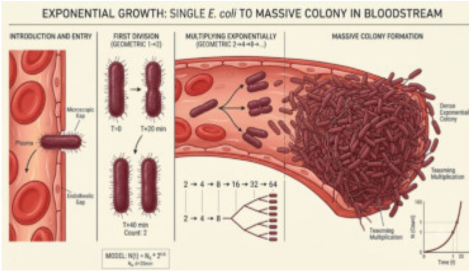
2-7
MATHEMATICS

Geometric Sequence (1)

Chapter 2 Sequences

Concept 2 · Geometric sequences

A single E. coli bacterium enters a bloodstream. Under ideal conditions, it does not just split into two; those two split into four, those four split into eight, doubling every 20 minutes. Within 24 hours, that single cell multiplies into a colony numbering in the sextillions—outnumbering the stars in the galaxy. This is the reality of geometric growth.



Guiding Question: How do processes with small starting values escalate rapidly, and how can we mathematically predict their cumulative impact over time?

Learning Outcomes

Calculate the general term of a geometric sequence, and determine certain terms in it.

Point!

❗ A sequence obtained by successively multiplying each preceding term by a constant number r is called **a geometric sequence** , and the constant number r is called **the common ratio** .

Example: A geometric sequence with the first term 1 and the common ratio 2

$1, \overset{\times 2}{} 2, \overset{\times 2}{} 4, \overset{\times 2}{} 8, \overset{\times 2}{} 16, \ldots$

❗ The general term of a geometric sequence with the first term a and the common ratio r is $a_n = ar^{n-1}$

$a, \quad a_2, \quad a_3, \quad \ldots, a_{n-1}, \quad a_n$

$\times r \quad \times r \quad \times r$

Multiplying a by r, $(n-1)$ times results in a_n

Warm Up

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The first term 2, the common ratio 3

(b) $5, 10, 20, 40, \ldots$

(c) $-4, 6, -9, \frac{27}{2}, \ldots$

2 The following sequence is a geometric sequence. Find the common ratio and the numbers that fit in the sequence below .

$A, 24, 12, B, \ldots$

Explanation

1 (a) $a_n = 2 \times 3^{n-1}$

> $5 \xrightarrow{\times r} 10 \xrightarrow{\times r} 20 \xrightarrow{\times r} 40, \ldots$

(b) Let r be the common ratio. Since the first term is 5 and the 2nd term is 10:

> $a_n = ar^{n-1}$

$5r = 10$

$r = 2$

Therefore, since it is a geometric sequence with the first term 5 and the common ratio 2:

$a_n = 5 \times 2^{n-1}$

(c) Let r be the common ratio. Since the first term is −4 and the 2nd term is 6:

$-4r = 6$

$r = -\dfrac{3}{2}$

Therefore, since it is a geometric sequence with the first term − 4 and the common ratio $-\dfrac{3}{2}$:

$a_n = -4 \times \left(-\dfrac{3}{2}\right)^{n-1}$

> Use parentheses ( ) when the common ratio is a negative number or a fraction.

2 Let r be the required common ratio.

$24r = 12 \qquad \therefore\ r = \dfrac{1}{2}$

> $A, \underset{\div\frac{1}{2}}{\overset{\times\frac{1}{2}}{\rightleftarrows}} 24, \xrightarrow{\times\frac{1}{2}} 12, \xrightarrow{\times\frac{1}{2}} B, \ldots$

Since **A** multiplied by $\dfrac{1}{2}$ becomes 24: $\qquad \therefore\ 24 \div \dfrac{1}{2} = 48$

Since 12 multiplied by $\dfrac{1}{2}$ becomes **B**: $\qquad \therefore\ 12 \times \dfrac{1}{2} = 6$

Therefore, Common ratio: $\dfrac{1}{2}$, **A**: 48, **B**: 6

Try

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The first term 3, the common ratio 4

(b) The first term 5, the common ratio -2

(c) The first term 8, the common ratio $-\frac{1}{3}$

(d) $2,\ 6,\ 18,\ 54,\ \dots$

(e) $-8,\ 12,\ -18,\ 27,\ \dots$

(f) $2,\ \frac{2}{3},\ \frac{2}{9},\ \frac{2}{27},\ \dots$

2 The following sequence is a geometric sequence. Find the common ratio and the numbers that fit in the sequence below .

$\mathrm{A},\ -18,\ 6,\ \mathrm{B},\ \dots$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The first term 5, the common ratio 2

(b) The first term 2, the common ratio 5

(c) The first term 7, the common ratio 4

(d) The first term 2, the common ratio -3

(e) The first term 3, the common ratio -5

(f) The first term 4, the common ratio -2

(g) The first term 4, the common ratio $\frac{1}{3}$

(h) The first term 3, the common ratio $-\frac{1}{2}$

(i) The first term 6, the common ratio $\frac{1}{2}$

(j) $3,\ 6,\ 12,\ 24,\ \dots$

(k) $2,\ -6,\ 18,\ -54,\ \dots$

(l) $5,\ -10,\ 20,\ -40,\ \dots$

(m) $48,\ 24,\ 12,\ 6,\ \dots$

(n) $\frac{1}{6},\ \frac{1}{4},\ \frac{3}{8},\ \frac{9}{16},\ \dots$

(o) $8,\ 16\sqrt{2},\ 64,\ 128\sqrt{2},\ \dots$

(p) $3,\ -\frac{15}{2},\ \frac{75}{4},\ -\frac{375}{8},\ \dots$

(q) $9,\ -3\sqrt{3},\ 3,\ -\sqrt{3},\ \dots$

(r) $5,\ -\frac{10}{3},\ \frac{20}{9},\ -\frac{40}{27},\ \dots$

2 The following sequences are geometric sequences. Find the common ratio and the numbers that fit in A and B.

(a) $\mathrm{A},\ -3,\ 9,\ \mathrm{B},\ \dots$ (b) $\mathrm{A},\ \frac{1}{3},\ \frac{1}{9},\ \mathrm{B},\ \dots$ (c) $\mathrm{A},\ -\frac{1}{4},\ \frac{1}{8},\ \mathrm{B},\ \dots$

★ Challenge

In an increasing geometric sequence, if $a_3 + a_5 = 80$ *and* $a_4 = 24$,then find a_7.

2-8

MATHEMATICS

Geometric Sequence (2)

Chapter 2 Sequences

Point!

! The general term of a geometric sequence with the first term a and the common ratio r is $a_n = \underline{ar^{n-1}}$

! When the sequence $a,\ b,\ c$ is a geometric sequence in this order: $\underline{b^2 = ac}$

Learning Outcomes

Calculate the general term of a geometric sequence, and determine certain terms in it.

Considering $a, \underbrace{\ }_{\times r} b, \underbrace{\ }_{\times r} c$: $b = ar,\ c = ar^2$

$b^2 = (ar)^2 = a^2r^2$ ......(i)

$ac = a \times ar^2 = a^2r^2$ ......(ii)

From (i) and (ii), $b^2 = ac$

Warm Up

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The 4th term is 24, the 6th term is 96

(b) The 3rd term is 12, the 6th term is −96

2 When the 3 numbers 2, x, 8 form a geometric sequence in this order, find the value of x

3 When the 3 numbers 6, a, b form an arithmetic sequence in this order, and $a,\ b,\ 16$ form a geometric sequence in this order, find the values of a and b

Explanation

1 (a) Let the general term be $a_n = ar^{n-1}$

Since the 4th term is 24, $ar^3 = 24$. ......(i)

$a_4 = ar^{4-1} = ar^3$

Since the 6th term is 96, $ar^5 = 96$. ......(ii)

$a_6 = ar^{6-1} = ar^5$

From (i) and (ii), $a \neq 0$, $r \neq 0$. Dividing (ii) by (i) gives:

$$\frac{ar^5}{ar^3} = \frac{96}{24}$$

Reduce the fraction.

$r^2 = 4$ Therefore, $r = \pm 2$

From (i), when $r = 2$, $8a = 24$, so $a = 3$

When $r = -2$, $-8a = 24$, so $a = -3$

Therefore, $\underline{a_n} = 3 \times 2^{n-1}$ or $\underline{a_n} = -3 \times (-2)^{n-1}$

(b) Let the general term be $a_n = ar^{n-1}$

Since the 3rd term is 12, $ar^2 = 12$. ......(i)

Since the 6th term is − 96, $ar^5 = -96$. ......(ii)

From (i) and (ii), $a \neq 0$, $r \neq 0$. Dividing (ii) by (i) gives:

$$\frac{ar^5}{ar^2} = \frac{-96}{12}$$

$r^3 = -8$

Since r is a real number, $r = -2$

Substitute this into (i): $4a = 12$, so $a = 3$

Therefore, $a_n = 3 \times (-2)^{n-1}$

2 Since $2, x, 8$ form a geometric sequence in this order, $x^2 = 16$

Solving this, $x = \pm 4$.

3 Since $6, a, b$ form an arithmetic sequence in this order, $2a = 6 + b$. ......(i)

Since $a, b, 16$ form a geometric sequence in this order, $b^2 = 16a$. ......(ii)

From (i), $b = 2a - 6$ ......(i)′. Substituting this into (ii) and simplifying: $4a^2 - 40a + 36 = 0$

Solving this, $a = 9, 1$

From (i)′, when $a = 9$, $b = 12$; when $a = 1$, $b = -4$

Therefore, $a = 9, b = 12$ **or** $a = 1, b = -4$

Try

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The 3rd term is 18, and the 5th term is 162

(b) The 2nd term is − 6, and the 5th term is 48

(c) The 3rd term is 12, and the 7th term is 192

2 When the 3 numbers $1, x, 12$ form a geometric sequence in this order, find the value of x.

3 When the 3 numbers $-2, a, b$ form an arithmetic sequence in this order, and $a, b, 18$ form a geometric sequence in this order, find the values of a and b.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the general term of the following geometric sequences $\{a_n\}$.

(a) The 2nd term is 6, and the 4th term is 54

(b) The 3rd term is 27, and the 5th term is 243

(c) The 3rd term is − 28, and the 5th term is − 112

(d) The 2nd term is 14, and the 5th term is 112

(e) The 4th term is 16, and the 7th term is − 128

(f) The 3rd term is − 20, and the 6th term is 160.

(g) The 2nd term is 3, and the 6th term is 768.

(h) The 3rd term is 9, and the 7th term is $\frac{729}{16}$.

(i) The 4th term is $\frac{2}{9}$, and the 8th term is 18.

2 When the 3 numbers 1, x, 25 form a geometric sequence in this order, find the value of x.

3 When the 3 numbers $\frac{1}{2}$, x, 8 form a geometric sequence in this order, find the value of x.

4 When the 3 numbers a, b, 2 form an arithmetic sequence in this order, and a, 2, b form a geometric sequence in this order, find the values of a and b.

★ Challenge

Budgeting for a Green Energy Initiative

A corporate sustainability department is planning a 3-year budget to phase out fossil fuels. They are tracking their yearly funding allocations across three consecutive stages, modeled under two different economic forecasts:

Model 1 (Linear Expansion): Under steady conditions, the funding for Year 1 is capped at −5 hundred thousand LE (initial infrastructure debt), followed by Year 2 (a) and Year 3 (b). These three numbers form an arithmetic sequence.

Model 2 (Compounding Scale): If a clean-energy breakthrough occurs, the budget shifts gears. The allocations for Year 2 (a), Year 3 (b), and a newly projected Year 4 budget of 45 hundred thousand LE form a geometric sequence.

Determine the exact budget values for a (Year 2) and b (Year 3) that satisfy both planning models simultaneously so the company can transition smoothly.

2-9

MATHEMATICS

Sum of Geometric Sequence (1)

Chapter 2 Sequences

Point!

! The sum S_n of a geometric sequence with the first term a and the common ratio r from the first term to the n–th term is:

When $r \neq 1$:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Formula to use when the common ratio r is less than 1

$$= \frac{a(r^n-1)}{r-1}$$

Formula to use when the common ratio r is greater than 1

Learning Outcomes

Calculate the sum of a geometric sequence of n terms.

Warm Up

Solve the following problems.

1 Find the sum S of the following geometric sequences.

(a) The first term 4, the common ratio 3, and the number of terms 5.

(b) The first term 8, the common ratio $\frac{1}{2}$, and the number of terms 6.

2 Find the sum S_n from the first term to the n-th term of the following geometric sequence.

$2, -6, 18, -54, \ldots$

Explanation

1 (a) Since $a = 4, r = 3, n = 5$:

Since the common ratio r is greater than 1, use $S_n = \frac{a(r^n-1)}{r-1}$.

$$S = \frac{4(3^5-1)}{3-1}$$

$$= 2(3^5-1) \qquad = 484$$

(b) Since $a = 8, r = \frac{1}{2}, n = 6$:

Since the common ratio r is less than 1, use $S_n = \frac{a(1-r^n)}{1-r}$.

$$S = \frac{8\{1-\left(\frac{1}{2}\right)^6\}}{1-\frac{1}{2}}$$

Multiply numerator and denominator by 2

$$= \frac{16\{1-\left(\frac{1}{2}\right)^6\}}{2-1}$$

$$= 16\{1-\left(\frac{1}{2}\right)^6\}$$

$$= \frac{63}{4}$$

2 Since $a = 2, r = -3$:

Since the common ratio r is less than 1, use $S_n = \frac{a(1-r^n)}{1-r}$.

$$S_n = \frac{2\{1-(-3)^n\}}{1-(-3)}$$

$$= \frac{1-(-3)^n}{2}$$

Try

Solve the following problems.

1 Find the sum S of the following geometric sequences.

(a) The first term 3, the common ratio 2, and the number of terms 5

(b) The first term 6, the common ratio − 2, and the number of terms 4

(c) The first term 6, the common ratio $\frac{1}{2}$, and the number of terms 6

2 Find the sum S_n from the first term to the n-th term of the following geometric sequences.

(a) The first term 3, the common ratio − 2

(b) $8,\ -4,\ 2,\ -1,\ \ldots$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the sum S of the following geometric sequences.

(a) The first term 5, the common ratio 3, and the number of terms 4

(b) The first term 1, the common ratio − 3, and the number of terms 4

(c) The first term 1, the common ratio $\frac{1}{3}$, and the number of terms 3

(d) The first term 3, the common ratio 2, and the number of terms 10

(e) The first term 2, the common ratio − 3, and the number of terms 6

(f) The first term 1, the common ratio $\frac{1}{2}$, and the number of terms 8

2 Find the sum S_n from the first term to the n-th term of the following geometric sequences.

(a) The first term 7, and the common ratio 2

(b) The first term 2, and the common ratio $\frac{1}{3}$

(c) $2,\ 6,\ 18,\ 54,\ \ldots$

(d) $1,\ -3,\ 9,\ -27,\ \ldots$

(e) The first term 4, and the common ratio 2

(f) The first term 16, and the common ratio $-\frac{1}{2}$

(g) $1,\ -2,\ 4,\ -8,\ \ldots$

★ Challenge

An environmental tech company designs sequential water filters where each stage traps a third of the remaining microplastics compared to the previous stage.

The filtration scale progresses geometrically: $3,\ 1,\ \frac{1}{3},\ \frac{1}{9},\ \ldots$ mg/L removed per stage.

Calculate the total mass of contaminants removed if the system uses 4 filtration stages in total.

2-10

MATHEMATICS

Sum of Geometric Sequence (2)

Chapter 2 Sequences

Learning Outcomes

Calculate the general term of a geometric sequence, find certain terms in it, and also the sum of all or some of its terms.

Warm Up

Find the first term and common ratio of the following geometric sequences.

1 The sum from the first term to the 3rd term is 35, the sum from the 3rd term to the 5th term is 140

2 The sum from the first term to the 3rd term is 21, the sum from the first term to the 6th term is 189

3 The 3rd term is 4, the sum from the first term to the 3rd term is 7

Explanation

1 Let a be the first term and r be the common ratio.

Since the sum from the first term to the 3rd term is 35, $a + ar + ar^2 = 35$. ……(i)

Since the sum from the 3rd term to the 5th term is 140, $ar^2 + ar^3 + ar^4 = 140$.. (ii)

From (ii), $r^2\left(a + ar + ar^2\right) = 140$

Transform into a form where (i) can be substituted.

Substitute (i) into this: $35r^2 = 140$

Solving this, $r = \pm 2$

When $r = 2$, from (i): $a + 2a + 4a = 35$

Solving this, $a = 5$

When $r = -2$, from (i): $a - 2a + 4a = 35$

Solving this, $a = \frac{35}{3}$

Therefore, First term: 5, Common ratio: 2 or First term: $\frac{35}{3}$, Common ratio: -2

2 Let a be the first term and r be the common ratio.

Since the sum from the first term to the 3rd term is 21, $a + ar + ar^2 = 21$. ……(i)

Since the sum from the first term to the 6th term is 189, $a + ar + ar^2 + ar^3 + ar^4 + ar^5 = 189$. ……(ii)

From (ii), $(a + ar + ar^2) + r^3\left(a + ar + ar^2\right) = 189$

Transform into a form where (i) can be substituted.

Substitute (i) into this: $21 + 21r^3 = 189$

Solving this in the real number range, $r = 2$

From (i), $a + 2a + 4a = 21$

Solving this, $a = 3$ Therefore, First term: 3, Common ratio: 2

3 Let a be the first term and r be the common ratio.

Since the 3rd term is 4, $ar^2 = 4$. ......(i)

Since the sum from the first term to the 3rd term is 7, $a + ar + ar^2 = 7$. ......(ii)

Since $r \neq 0$, from (i), $a = \dfrac{4}{r^2}$

Substitute this into (ii): $\dfrac{4}{r^2} + \dfrac{4}{r^2} \times r + \dfrac{4}{r^2} \times r^2 = 7$

Multiply both sides by r^2

$4 + 4r + 4r^2 = 7r^2$

Solving this, $r = 2, -\dfrac{2}{3}$

When $r = 2$, from (i), $4a = 4$

Solving this, $a = 1$

$r = -\dfrac{2}{3}$, from (i), $\dfrac{4}{9}a = 4$

Solving this, $a = 9$

Therefore, First term: 1, Common ratio: 2 or First term: 9, Common ratio: $-\dfrac{2}{3}$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Find the first term and common ratio of the following geometric sequences.

1 The sum from the first to the 3rd term is 21, and the sum from the 3rd to the 5th term is 84

2 The sum from the first to the 3rd term is 26, and the sum from the first to the 6th term is 728

3 The 2nd term is 3, and the sum from the first to the 3rd term is 13

Exercise

Find the first term and common ratio of the following geometric sequences.

1 The sum from the first to the 3rd term is 9, and the sum from the 4th to the 6th term is -72

2 The sum from the first to the 3rd term is 9, and the sum from the 3rd to the 5th term is 36

3 The sum from the first to the 3rd term is 78, and the sum from the 3rd to the 5th term is 702

4 The sum of the first and the 2nd terms is 3, and the sum from the first to the 4th term is 51

5 The sum from the first to the 3rd term is 65, and the sum from the first to the 6th term is 1820

6 The sum of the first and the 2nd terms is − 8, and the sum from the first to the 4th term is − 80

7 The 2nd term is 6, and the sum from the first to the 3rd term is 21

8 The 3rd term is 12, and the sum from the first to the 3rd term is 9

9 The 3rd term is 20, and the sum from the first to the 3rd term is 35

★ Challenge

If S_n is the sum of the first n terms of the geometric sequence such that $S_8 = 255,\ S_6 = 63\ and\ a_8 - a_7 = 64$, find a_{10}.

Reflect

How does changing the common ratio r from a value greater than 1 to a value between 0 and 1 fundamentally alter the real-world behavior modeled by a geometric sequence?

2-11

MATHEMATICS

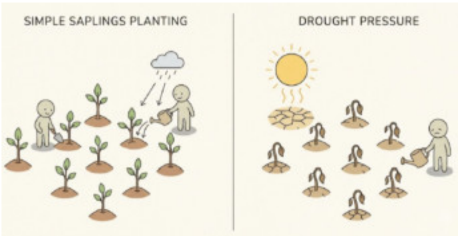
Summation Symbol Σ (1)

Chapter 2 Sequences

Learning Outcomes

Use the summation symbol to represent mathematical sequences and calculate their values in different sequences.

Concept 3 · Summation



In environmental biology, forest recovery follows a dual-force mathematical pattern. Following a wildfire, conservationists plant 200 saplings. To accelerate restoration, they scale up efforts, adding 100 more saplings to the quota each consecutive year (a linear, arithmetic progression: 200, 300, 400...). Concurrently, natural pressures like drought take a toll: only 60% of the trees survive from one year to the next (an exponential, geometric decay with a ratio of 0.6).

Calculating the total surviving population after a decade requires tracking how each year's expanding planting cycle continuously decays. Mathematics simplifies this overlapping process through summation notation, providing a unified framework to analyze simultaneous linear growth and exponential decay.

Guiding Question: How can we simplify operations on sequences, such as multiplying terms or adding nested sums, to find a clear algebraic formula?

Point!

! For a sequence where the n—th term (general term) is a_n, the sum from the 1st term to the n-th term is expressed as

$$\sum_{k=1}^{n} a_k$$

Example: $\sum_{k=1}^{4} 7k = 7 + 14 + 21 + 28$

The sum from the first to the 4th term of a sequence with the general term $7n$.

Example: The sum from the first to the 8th term of a sequence with the n-th term (general term) $3n$: $\sum_{k=1}^{8} 3k$

Change n to k.

Warm Up

Solve the following problems.

1 Write the sums of the following sequences in the form of a sum of terms without using Σ.

(a) $\sum_{k=1}^{5} 2k$ (b) $\sum_{k=1}^{3} (k^2 - 3)$ (c) $\sum_{k=1}^{4} 3^k$ (d) $\sum_{k=1}^{3} 2$

2 Express the following expressions using the summation symbol Σ.

(a) $3 + 7 + 11 + \dots + (4n - 1)$ (b) $6^2 + 7^2 + 8^2 + \dots + 15^2$

Explanation

1

(a) Substitute $k = 1,\ 2,\ 3,\ 4,\ 5$ into $2k$ and write them as terms in a sum.

$$\sum_{k=1}^{5} 2k = \underline{2+4+6+8+10}$$

(b) Substitute $k = 1,\ 2,\ 3$ into k^2-3 and write them as terms in a sum.

$$\sum_{k=1}^{3} (k^2-3) = \underline{-2+1+6}$$

(c) Substitute $k = 1, 2, 3, 4$ into 3^k and write them as terms in a sum.

$$\sum_{k=1}^{4} 3^k = \underline{3+9+27+81}$$

(d) The sequence where the n-th term is represented by 2 is 2, 2, 2, ...

$$\sum_{k=1}^{3} 2 = \underline{2+2+2}$$

2

(a) For problems where the last term is expressed as an expression in terms of n, just change n to k and write it inside Σ.

(Below Σ is $k = 1$, above is n)

$$\sum_{k=1}^{n} (4k-1)$$

(b) To express with the symbol Σ, you need the general term a_n and the index of the last term.

First, find the general term. The general term is $a_n = (n+5)^2$.

Find the index of the last term. 15^2 is the 10th term.

Therefore,

$$\sum_{k=1}^{10} (k+5)^2$$

Try

Solve the following problems.

1 Write the sums of the following sequences in the form of a sum of terms without using Σ.

(a) $\displaystyle\sum_{k=1}^{5} (2k+1)$ (b) $\displaystyle\sum_{k=1}^{3} (3k^2+2)$ (c) $\displaystyle\sum_{k=1}^{4} 2^k$ (d) $\displaystyle\sum_{k=1}^{6} 3$

2 Express the following expressions using the summation symbol Σ.

(a) $5+8+11+\ldots+(3n+2)$

(b) The sum from the 1st to the n-th term of the sequence $a_n = 2n+1$

(c) $4^2+5^2+6^2+\ldots+13^2$

(d) $2+5+8+\ldots+29$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Write the sums of the following sequences in the form of a sum of terms without using Σ.

(a) $\sum_{k=1}^{5} 3k$ (b) $\sum_{k=1}^{4} (4k-2)$ (c) $\sum_{k=1}^{5} (2k-3)$ (d) $\sum_{k=1}^{4} k^2$

(e) $\sum_{k=1}^{3} (2k^2+3)$ (f) $\sum_{k=1}^{4} (k^2-2k)$ (g) $\sum_{k=1}^{3} 3^k$ (h) $\sum_{k=1}^{3} (5^k+3)$

(i) $\sum_{k=1}^{4} (2^k-3)$ (j) $\sum_{k=1}^{4} 5$ (k) $\sum_{k=1}^{3} 3$ (l) $\sum_{k=1}^{5} 4$

2 Express the following expressions using the summation symbol Σ.

(a) $1 + 3 + 5 + \dots + (2n-1)$

(b) $1 + 7 + 25 + 79 + \dots + (3^n-2)$

(c) $1 + 2 + 4 + 8 + \dots + 2^{n-1}$

(d) The sum from the 1st to the n-th term of sequence $a_n = 7n-1$

(e) The sum from the 1st to the n-th term of sequence $a_n = 4n+5$

(f) The sum from the 1st to the n-th term of sequence $a_n = 3n-2$

(g) $7^2+8^2+9^2+\dots+21^2$

(h) $12^2+13^2+14^2+\dots+18^2$

(i) $10^2+11^2+12^2+\dots+20^2$

(j) $8 + 16 + 24 + \dots + 56$

(k) $7 + 13 + 19 + \dots + 73$

(l) $4 + 9 + 14 + \dots + 119$

★ Challenge

A green-tech startup is expanding its network of solar-powered charging stations across a metropolitan area. A startup installs 1 solar panel in (Week 1), 4 in (Week 2), 7 in (Week 3), and (3n - 2) panels in (Week n). Write a single mathematical expression using the summation symbol Σ to represent the total cumulative panels installed from Week 1 through Week n.

2-12

MATHEMATICS

Summation Symbol Σ (2)

Chapter 2 Sequences

Learning Outcomes

Use the summation symbol to represent mathematical sequences and calculate their values in different sequences.

Point!

! Formulas for Summation Symbol Σ

$\sum_{k=1}^{n} c = \underline{nc}$ (c is a constant) $\quad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

! Properties of Summation Symbol Σ (where p is a constant independent of k)

$\sum_{k=1}^{n} pa_k = p\sum_{k=1}^{n} a_k$ $\quad$ Example: $\sum_{k=1}^{n} 5k = 5\sum_{k=1}^{n} k$

$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ $\quad$ Example: $\sum_{k=1}^{n} (k^2+4) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 4$

$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$ $\quad$ Example: $\sum_{k=1}^{n} (2k-3) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 3$

Warm Up

Find the following sums.

(a) $\sum_{k=1}^{n} (-3)$

(b) $\sum_{k=1}^{n} (5k+4)$

(c) $1\times 3 + 2\times 8 + 3\times 13 + \ldots + n(5n-2)$

(d) $\sum_{k=1}^{4} (2k-1)$

Explanation

(a) $\sum_{k=1}^{n} (-3)$

$= -3n$

(b) $\sum_{k=1}^{n}(5k+4)$

$= 5\sum_{k=1}^{n} k + \sum_{k=1}^{n} 4$

$= 5 \times \frac{1}{2}n(n+1) + 4n$

Do not expand the parentheses; factor out the fraction and common factor.

$= \frac{1}{2}n\{5(n+1)+8\} = \frac{1}{2}n(5n+13)$

(c) $1 \times 3 + 2 \times 8 + 3 \times 13 + \ldots + n(5n-2)$

Write using summation notation Σ

$= \sum_{k=1}^{n} k(5k-2)$

Expand the expression inside the summation.

$= \sum_{k=1}^{n}(5k^2-2k)$

$= 5\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$

$= 5 \times \frac{1}{6}n(n+1)(2n+1) - 2 \times \frac{1}{2}n(n+1)$

$= \frac{5}{6}n(n+1)(2n+1) - n(n+1)$

Do not expand the parentheses; factor out the fraction and common factor.

$= \frac{1}{6}n(n+1)\{5(2n+1)-6\}$

$= \frac{1}{6}n(n+1)(10n-1)$

(d) $\sum_{k=1}^{4}(2k-1)$

$= 2\sum_{k=1}^{4} k - \sum_{k=1}^{4} 1$

Substitute 4 for n in the formula.

$= 2 \times \frac{1}{2} \times 4(4+1) - 4$

$= 16$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Find the following sums.

(a) $\sum_{k=1}^{n} 3$

(b) $\sum_{k=1}^{n}(2k-3)$

(c) $\sum_{k=1}^{n}(k-1)(3k-1)$

(d) $1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots + n(2n+1)$

(e) $\sum_{k=1}^{5}(3k-2)$

Exercise

Find the following sums.

(a) $\sum_{k=1}^{n} 6$

(b) $\sum_{k=1}^{n} (-4)$

(c) $\sum_{k=1}^{n} 1$

(d) $\sum_{k=1}^{n} (4k-1)$

(e) $\sum_{k=1}^{n} (3k+2)$

(f) $\sum_{k=1}^{n} (6k-3)$

(g) $\sum_{k=1}^{n} k(k-2)$

(h) $\sum_{k=1}^{n} k(2k-2)$

(i) $\sum_{k=1}^{n} (k-1)^2$

(j) $1 \times 4 + 2 \times 5 + 3 \times 6 + \ldots + n(n+3)$

(k) $1 \times 5 + 2 \times 8 + 3 \times 11 + \ldots + n(3n+2)$

(l) $2 \times 5 + 3 \times 6 + 4 \times 7 + \ldots + (n+1)(n+4)$

(m) $\sum_{k=1}^{20} (2k-7)$

(n) $\sum_{k=1}^{8} (4k+5)$

(o) $\sum_{k=1}^{10} (2k-1)$

★ Challenge

Find the following sum:

$$\sum_{r=4}^{8} (r^2 + r + 5)$$

2-13

MATHEMATICS

Summation Symbol Σ (3)

Chapter 2 Sequences

Point!

! Formulas for Summation Symbol Σ

$\sum_{k=1}^{n} k^3 = \{\frac{1}{2}n(n+1)\}^2$

! Sum of geometric sequence

$\sum_{k=1}^{n} ar^{k-1}$ is the sum from the first term to the n—th term of a geometric sequence with the first term a and the common ratio r, so:

When $r<1$, $\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$ When $r>1$, $\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n-1)}{r-1}$

! A sequence expressed as $a_n = r^{n-1}$ can be rewritten as $a_n = 1 \times r^{n-1}$, so it is a geometric sequence with the first term 1 and the common ratio r.

Example: $a_n = 4^{n-1}$

$= 1 \times 4^{n-1}$ First term: 1, Common ratio: 4

! A sequence expressed as $a_n = r^n$ can be rewritten as $a_n = r \times r^{n-1}$, so it is a geometric sequence with the first term r and the common ratio r.

Example: $a_n = 4^n$

$= 4 \times 4^{n-1}$ First term: 4, Common ratio: 4

Learning Outcomes

Use the summation symbol to represent mathematical sequences and calculate their values in different sequences.

Remember as "the square of $\sum_{k=1}^{n} k$".

Warm Up

Find the following sums.

(a) $\sum_{k=1}^{n} k^2(k+1)$ (b) $\sum_{k=1}^{n} 2\times(-3)^{k-1}$ **(c)** $\sum_{k=1}^{n} 4^k$

Explanation

(a) $\sum_{k=1}^{n} k^2(k+1)$

Expand the expression inside the summation.

$= \sum_{k=1}^{n} (k^3+k^2)$

$= \{\frac{1}{2}n(n+1)\}^2 + \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)$

Do not expand the parentheses; factor out the fraction and common factor.

$= \frac{1}{12}n(n+1)\{3n(n+1)+2(2n+1)\}$

$$= \frac{1}{12}n(n+1)(3n^2+7n+2)$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+1)$$

Factorise if possible.

(b) $\sum_{k=1}^{n} 2\times(-3)^{k-1}$

$$= \frac{2\{1-(-3)^n\}}{1-(-3)}$$

$$= \frac{2\{1-(-3)^n\}}{4}$$

$$= \frac{1-(-3)^n}{2}$$

First term: 2, Common ratio: − 3

(c) $\sum_{k=1}^{n} 4^k$

$$= \sum_{k=1}^{n} 4\times 4^{k-1}$$

$$= \frac{4(4^n-1)}{4-1}$$

$$= \frac{4(4^n-1)}{3}$$

Write in the form ar^{n-1}.

First term: 4, Common ratio: 4

Try

Find the following sums.

(a) $\sum_{k=1}^{n} k^2(k-2)$ **(b)** $\sum_{k=1}^{n} 3\times(-2)^{k-1}$

(c) $\sum_{k=1}^{n} 4^{k-1}$ **(d)** $\sum_{k=1}^{n} (-5)^k$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the following sums.

(a) $\sum_{k=1}^{n} k(k^2-1)$ **(b)** $\sum_{k=1}^{n} k^2(k+2)$

(c) $\sum_{k=1}^{n} k(k^2+2)$ **(d)** $\sum_{k=1}^{n} k^2(k-3)$

(e) $\sum_{k=1}^{n} 6\times 3^{k-1}$ **(f)** $\sum_{k=1}^{n} 2\times(-1)^{k-1}$

(g) $\sum_{k=1}^{n} 4\times 2^{k-1}$ **(h)** $\sum_{k=1}^{n} 8\times(-3)^{k-1}$

(i) $\sum_{k=1}^{n} 3^{k-1}$ **(j)** $\sum_{k=1}^{4} (-2)^{k-1}$

(k) $\sum_{k=1}^{n} (-2)^{k-1}$ **(l)** $\sum_{k=1}^{3} 3^{k-1}$

(m) $\sum_{k=1}^{n} (-3)^k$ **(n)** $\sum_{k=1}^{5} 2^k$

(o) $\sum_{k=1}^{n} 2^k$ **(p)** $\sum_{k=1}^{4} (-3)^k$

★ Challenge

Find the following sum:

$$\sum_{r=5}^{9} (2r+7)$$

2-14

MATHEMATICS

Sum of Fractional Sequences

Chapter 2 Sequences

Learning Outcomes

Calculate the sum of fractional sequences using fraction decomposition.

Point!

! Steps to Find the Sum of Fractional Sequences.

❶ Transform each term using the following formula.

$$\frac{1}{ab}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)\qquad \text{Example: } \frac{1}{4\times 10}=\frac{1}{6}\left(\frac{1}{4}-\frac{1}{10}\right)$$

❷ Cancel out the terms that sum to zero and calculate the remaining parts.

For sequences containing √ in the denominator, first **rationalize the denominator** .

When the sum of a sequence containing fractions or √ is expressed using summation notation Σ, rewrite it into **the form of a sum of terms** .

Warm Up

Find the following sums.

(a) $\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots+\frac{1}{(3n-1)(3n+2)}$ **(b)** $\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}$

Explanation

(a) $\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots+\frac{1}{(3n-1)(3n+2)}$

$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)+\frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)+\dots+\frac{1}{3}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)$

$=\frac{1}{3}\left\{\left(\frac{1}{2}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{8}}\right)+\left(\cancel{\frac{1}{8}}-\cancel{\frac{1}{11}}\right)+\dots+\left(\cancel{\frac{1}{3n-1}}-\frac{1}{3n+2}\right)\right\}$

$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)$

$=\frac{n}{2(3n+2)}$

Transform each term using the formula.

Factor out $\frac{1}{3}$

Cancel out the terms that sum to zero and calculate the remaining parts.

(b) $\sum_{k=1}^{n} \frac{1}{\sqrt{k+2}+\sqrt{k+1}}$

First rationalize the denominator.

$= \sum_{k=1}^{n} \frac{\sqrt{k+2}-\sqrt{k+1}}{\left(\sqrt{k+2}+\sqrt{k+1}\right)\left(\sqrt{k+2}-\sqrt{k+1}\right)}$

$= \sum_{k=1}^{n} \frac{\sqrt{k+2}-\sqrt{k+1}}{(k+2)-(k+1)}$

$= \sum_{k=1}^{n} \left(\sqrt{k+2}-\sqrt{k+1}\right)$

Rewrite in the form of a sum of terms.

$= (\cancel{\sqrt{3}} - \sqrt{2}) + (\cancel{\sqrt{4}} - \cancel{\sqrt{3}}) + (\cancel{\sqrt{5}} - \cancel{\sqrt{4}}) + (\cancel{\sqrt{6}} - \cancel{\sqrt{5}}) + \ldots + (\sqrt{n+2} - \sqrt{n+1})$

Cancel out the terms that sum to zero and calculate the remaining parts.

$= \sqrt{n+2} - \sqrt{2}$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Find the following sums.

(a) $\frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \ldots + \frac{1}{(n+1)(n+2)}$

(b) $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots + \frac{1}{(2n-1)(2n+1)}$

(c) $\frac{1}{4\times6} + \frac{1}{5\times7} + \frac{1}{6\times8} + \ldots + \frac{1}{(n+3)(n+5)}$

(d) $\sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$

Exercise

Solve the following problems.

1 Find the following sums.

(a) $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{n(n+1)}$

(b) $\frac{1}{2\times4} + \frac{1}{4\times6} + \frac{1}{6\times8} + \ldots + \frac{1}{2n(2n+2)}$

(c) $\frac{1}{1\times3} + \frac{1}{2\times4} + \frac{1}{3\times5} + \ldots + \frac{1}{n(n+2)}$

(d) $\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}}$

(e) $\frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \ldots + \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

2 Find the following sums.

(a) $\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}+\dots+\frac{1}{(n+2)(n+3)}$

(b) $\frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\dots+\frac{1}{(3n-2)(3n+1)}$

(c) $\frac{1}{1\times7}+\frac{1}{4\times10}+\frac{1}{7\times13}+\dots+\frac{1}{(3n-2)(3n+4)}$

(d) $\sum_{k=1}^{n}\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}$

(e) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\dots+\frac{1}{\sqrt{n}+\sqrt{n+1}}$

★ Challenge

Find with steps

$$\sum_{r=1}^{30}\frac{1}{r(r+2)}$$

2-15 MATHEMATICS

Sum of (Arithmetic Sequence)×(Geometric Sequence)

Chapter 2 Sequences

Point!

! For the sum S of (an arithmetic sequence) × (a geometric sequence with the common ratio r), use **$S-rS$**

Warm Up

Find the following sum S

$S = 3 \times 1 + 5 \times 3 + 7 \times 3^2 + 9 \times 3^3 + \ldots + (2n+1) \times 3^{n-1}$

Explanation

The sum to be found is the sum of (Arithmetic sequence with first term 3, the common difference 2) × (Geometric sequence with the first term 1, the common ratio 3).

$$S = 3 \times 1 + 5 \times 3 + 7 \times 3^2 + 9 \times 3^3 + \ldots + (2n+1) \times 3^{n-1}$$

Multiply both sides by the common ratio 3 of the geometric sequence.

$$3S = 3 \times 3 + 5 \times 3^2 + 7 \times 3^3 + 9 \times 3^4 + \ldots + (2n+1) \times 3^n$$

Subtract the sides of the two equations.

$$\begin{array}{rl} S = & 3 \times 1 + 5 \times 3 + 7 \times 3^2 + 9 \times 3^3 + \ldots + (2n+1) \times 3^{n-1} \\ -)\ 3S = & \qquad 3 \times 3 + 5 \times 3^2 + 7 \times 3^3 + \ldots + (2n-1) \times 3^{n-1} + (2n+1) \times 3^n \\ \hline -2S = & 3 \times 1 + 2 \times 3 + 2 \times 3^2 + 2 \times 3^3 + \ldots + 2 \times 3^{n-1} \qquad - (2n+1) \times 3^n \end{array}$$

$$-2S = 3 + \frac{2(3^n - 1)}{3 - 1} - 2 - (2n+1) \times 3^n$$

$$-2S = 3^n - (2n+1) \times 3^n$$

$$-2S = -2n \times 3^n$$

$$S = \underline{n \times 3^n}$$

Try

Find the following sum S

$S = 2 \times 1 + 3 \times 3 + 4 \times 3^2 + 5 \times 3^3 + \ldots + (n+1) \times 3^{n-1}$

Learning Outcomes

Find the sum of the product of an arithmetic sequence and a geometric sequence.

Common Mistakes

$S = 3 \times 1 + 5 \times 3 + 7 \times 3^2 + 9 \times 3^3 + \ldots + (2n+1) \times 3^{n-1}$

$-)3S = 3 \times 3 + 5 \times 3^2 + 7 \times 3^3 + \ldots + (2n-1) \times 3^{n-1} + (2n+1) \times 3^n$

$-2S = 3 \times 1 + 2 \times 3 + 2 \times 3^2 + 2 \times 3^3 + \ldots + 2 \times 3^{n-1} + (2n+1) \times 3^n$

You forgot to change the sign of the last term to minus when subtracting $3S$ from S.

Use $S - rS$.

Write the equations so that the powers of the common ratio align vertically.

Supplement terms to make subtraction easier.

The sum from the 2nd term to the n-th term of a geometric sequence with first term 2, common ratio 3. (The sum from the first term to the n-th term The first term)

Divide both sides by -2 Watch the '-' sign.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the following sums S

1 $S = 1\times1 + 2\times2 + 3\times2^2 + 4\times2^3 + \ldots + n\times2^{n-1}$

2 $S = 1\times3 + 3\times3^2 + 5\times3^3 + 7\times3^4 + \ldots + (2n-1)\times3^n$

★ Challenge

Find the following sum 5 , 5 + 9 , 5 + 9 + 13 , 5 + 9 + 13 + 17,………… + (4k + 1)

2-16
MATHEMATICS

Sequence where General Term is a Sum

Chapter 2 Sequences

Point!

! Calculate the k-th term of a sequence where each term is expressed as a sum using summation notation Σ.

Learning Outcomes

Deduce the sum of a sequence when its general term is expressed as the sum of some of its terms, and connect the sum of the sequence with its general term.

Warm Up

Solve the following problems for the sequence 3, 3+5, 3+5+7, 3+5+7+9, ...

(a) Find the k-th term.

(b) Find the sum from the first term to the n-th term.

Explanation

(a) The k–th term of this sequence is:

$$a_k = 3+5+7+\ldots+(2k+1)$$
$$= \sum_{i=1}^{k}(2i+1)$$
$$= 2\times\frac{1}{2}k(k+1)+k$$

Calculating this gives $a_k = \underline{\boldsymbol{k(k+2)}}$

(b) From **(a)**, the sum S_n from the first term to the n–th term is:

$$S_n = \sum_{k=1}^{n} k(k+2)$$
$$= \sum_{k=1}^{n}(k^2+2k) \quad = \frac{1}{6}n(n+1)(2n+1) + 2\times\frac{1}{2}n(n+1)$$

Calculating this gives $S_n = \dfrac{1}{6}n(n+1)(2n+7)$

Create a formula considering the regularity.

Calculate using summation notation Σ.

Since it is the sum from the first term to the k–th term, we use the variable i.

Put a_k inside Σ. $S_n = \sum_{k=1}^{n} a_k$

Try

Solve the following problems for the sequence 1, 1+3, 1+3+5, 1+3+5+7, ...

1 Find the k-th term.

2 Find the sum from the first term to the n-th term.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Solve the following problems for the sequence $1,\ 1+5,\ 1+5+9,\ 1+5+9+13,\ \ldots$

(a) Find the k-th term.

(b) Find the sum from the first term to the n-th term.

2 Solve the following problems for the sequence $2,\ 2+4,\ 2+4+6,\ 2+4+6+8,\ \ldots$

(a) Find the k-th term.

(b) Find the sum from the first term to the n—th term.

★ Challenge

Find the general term of the sequence $\{a_n\}$ where the sum of the first n terms is given by the formula $S_n = 2n^2 + 5n$

2-17

MATHEMATICS

Sum of Sequence and General Term

Chapter 2 Sequences

Point!

! Let S_n be the sum from the first term a_1 to the n−th term of the sequence $\{a_n\}$.

When $\boldsymbol{n} \geq \mathbf{2}$, $a_n = \boldsymbol{S_n - S_{n-1}}$.

The first term is $a_1 = \boldsymbol{S_1}$.

! Check if an obtained under the condition $n \geq 2$ also holds for the first term by substituting $n = 1$ into the formula for a_n.

If it holds ⇒ Keep a_n as is.

If it does not hold ⇒ Write both a_1 and a_n.

Example: If a_n found by $S_n - S_{n-1}$ is $a_n = 2n^2 - 2n + 1$ …… (i),

and a_1 found from S_1 is $a_1 = 0$ ……(ii), substituting $n = 1$ into (i) gives $a_1 = 1$, which does not hold.

Therefore, the general term is $\boldsymbol{a_1 = 0}$**, and** $\boldsymbol{a_n = 2n^2 - 2n + 1}$ **when** $\boldsymbol{n \geq 2}$.

$$\underbrace{\underbrace{a_1 + a_2 + a_3 + \dots + a_{n-1}}_{S_{n-1}} + a_n}_{S_n}$$

Learning Outcomes

Deduce the sum of a sequence when its general term is expressed as the sum of some of its terms, and connect the sum of the sequence with its general term.

Warm Up

Find the general term of the sequence $\{a_n\}$ where the sum Sn from the first term to the n−th term is given by the following formulas.

(a) $S_n = n^2 - 3n$ **(b)** $S_n = 5^n - 1$

Explanation

(a) When $n \geq 2$,

Deduction if not written.

$a_n = S_n - S_{n-1}$

$S_{n-1} = (n-1)^2 - 3(n-1)$

$= (n^2 - 3n) - \{(n-1)^2 - 3(n-1)\} \qquad = 2n - 4$ …… (i)

The first term is $a_1 = S_1$

Deduction if not written.

$= 1^2 - 3 \times 1 \qquad = -2$ …… (ii)

From (i) and (ii), $a_n = 2n - 4$ also holds when $n = 1$

Deduction if not written.

Check by substituting $n = 1$ into formula (i).

Therefore, the general term is $\boldsymbol{a_n = 2n - 4}$

Common Mistakes

$a_n = S_n - S_{n-1}$

$= (n^2 - 3n) - \{(n-1)^2 - 3(n-1)\} = 2n - 4$

Therefore, the general term is $\boldsymbol{a_n = 2n - 4}$

You overlooked the prerequisite $n \geq 2$ and failed to perform the final verification for $n = 1$

(b) When $n \geq 2$,

$a_n = S_n - S_{n-1}$

$= (5^n - 1) - (5^{n-1} - 1)$

$= 5^n - 5^{n-1}$

$= 5 \times 5^{n-1} - 1 \times 5^{n-1}$

$= 4 \times 5^{n-1}$ ...... (i)

The first term is $a_1 = S_1$

$= 5^1 - 1 \qquad = 4$ ...... (*ii*)

From (i) and (ii), $a_n = 4 \times 5^{n-1}$ also holds when $n = 1$

Therefore, the general term is $a_n = 4 \times 5^{n-1}$

Deduction if not written.

$5^n = 5^1 \times 5^{n-1}$

Deduction if not written.

Deduction if not written.

Try

Find the general term of the sequence $\{a_n\}$ where the sum S_n from the first term to the n-th term is given by the following formulas.

1. $S_n = 2n^2 + 5n$
2. $S_n = 3^n - 1$
3. $S_n = n^3 + 2n + 6$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the general term of the sequence $\{a_n\}$ where the sum S_n from the first term to the n-th term is given by the following formulas.

1. $S_n = n^2$
2. $S_n = n^2 - 4n$
3. $S_n = n^2 + 2n - 1$
4. $S_n = 2^n - 1$
5. $S_n = 5^n - 2$
6. $S_n = 7^n - 1$
7. $S_n = n^3 - 1$
8. $S_n = 2^{n+2} - 3$

★ Challenge

Find the general term of the sequence $\{a_n\}$ where the sum S_n from the first term to the n–th term is given by the following formulas.

$$S_n = 3^{n+1} - 2n + 1$$

Reflect

How did breaking down a complex, combined sequence (AGS) into its simpler parts (arithmetic and geometric components) change your perspective on tackling intimidating multi-step problems in life or science?

2-18 MATHEMATICS

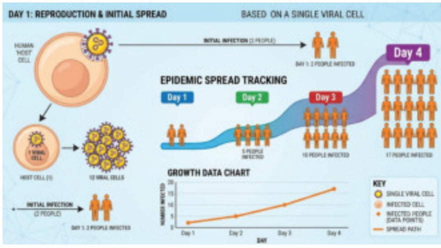
Difference Sequences

Chapter 2 Sequences

Learning Outcomes

Finds the general term using difference sequences.

Concept 4 · Advanced sequences



Consider how a single viral cell reproduces inside a host, or how a public health department tracks the spread of an epidemic. On Day 1, maybe 2 people are infected. On Day 2, 5 people. On Day 3, 10 people. On Day 4, 17 people. If you look closely at the changes day by day, the virus isn't spreading at a constant speed, but the acceleration of the spread follows a predictable, hidden pattern. By analyzing these layers of changes, epidemiologists can predict peak infection rates and allocate hospital resources effectively.

Guiding Question: How can we discover a hidden, constant pattern in a sequence of numbers when the initial differences between them keep changing?

Point!

! A sequence formed by the differences between adjacent terms is called **a difference sequence** .

Example: The difference sequence of 10, 8, 4, −2, −10, ... is −2, −4, −6, −8, ...

! For sequences where the general term is difficult to estimate, it can sometimes be found from the first term and the difference sequence.

When $n \geq 2$, $\quad a_n = a_1 + \sum_{k=1}^{n-1} b_k$

Check if it holds for $n = 1$ by substituting $n = 1$ into the formula for a_n and seeing if it equals the given a_1.

$a_1,\ a_2,\ a_3,\ \ldots,\ a_{n-1},\ a_n$

$b_1\quad b_2\quad \ldots\quad b_{n-1}$

$a_n = a_1 + b_1 + b_2 + \ldots + b_{n-1}$

Warm Up

Find the general term of the following sequences $\{a_n\}$.

1 $2,\ 5,\ 12,\ 23,\ 38,\ 57,\ \ldots$

2 $2,\ 3,\ 6,\ 15,\ 42,\ 123,\ \ldots$

Explanation

1 Let $\{b_n\}$ be the difference sequence of the given sequence.

$\{a_n\}: 2, 5, 12, 23, 38, 57, \ldots$

$\{b_n\}: 3, 7, 11, 15, 19, \ldots$

Since $\{b_n\}$ is an arithmetic sequence with the first term 3 and the common difference 4:

$b_n = 3 + (n-1) \times 4$

$= 4n - 1$

Therefore, when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

$$= 2 + \sum_{k=1}^{n-1} (4k-1) \qquad = 2 + 4 \times \frac{1}{2}(n-1)\{(n-1)+1\} - (n-1)$$

$$= 2 + 2n(n-1) - (n-1) \qquad = 2 + 2n^2 - 2n - n + 1 \qquad = 2n^2 - 3n + 3$$

This formula also holds when $n = 1$

Therefore, $a_n = 2n^2 - 3n + 3$

Common Mistakes

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

$$= 2 + \sum_{k=1}^{n-1} (4k-1)$$

$$= 2 + 4 \times \frac{1}{2} \times n(n+1) - n$$

$$= 2 + 2n^2 + 2n - n$$

$$= 2n^2 + n + 2$$

You missed replacing all instances of n with $n-1$ in the formula.

Deduction if not written.

Deduction if not written.

Use the formula noting that it is the sum up to the $(n-1)$-th term.

$$\sum_{k=1}^{n} (4k-1) = 4 \times \frac{1}{2} n(n+1) - n$$

Deduction if not written.

Be sure to substitute $n = 1$ into the derived formula for a_n to confirm it equals the given a_1. $a1 = 2 \times 1^2 - 3 \times 1 + 3 = 2$

2 Let $\{b_n\}$ be the difference sequence of the given sequence.

Deduction if not written.

$\{a_n\}: 2, 3, 6, 15, 42, 123, \ldots$

$\{b_n\}: 1, 3, 9, 27, 81, \ldots$

Since $\{b_n\}$ is a geometric sequence with the first term 1 and the common ratio 3:

This form is fine since Σ will be used.

$b_n = 1 \times 3^{n-1}$

Deduction if not written.

Therefore, when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

$$= 2 + \sum_{k=1}^{n-1} 1 \times 3^{k-1} = 2 + \frac{3^{n-1} - 1}{3 - 1} = \frac{4 + 3^{n-1} - 1}{2} = \frac{3^{n-1} + 3}{2}$$

Use the formula noting that it is the sum up to the $(n-1)$-th term.

$\sum_{k=1}^{n} 1 \times 3^{k-1} = \frac{3^n - 1}{3 - 1}$

This formula also holds when $n = 1$

Deduction if not written.

Therefore, $a_n = \frac{3^{n-1}+3}{2}$

Try

Find the general term of the following sequences $\{a_n\}$.

(a) $2, 10, 24, 44, 70, 102, \ldots$ **(b)** $5, 7, 11, 19, 35, \ldots$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the general term of the following sequences $\{a_n\}$.

(a) $1, 2, 5, 14, 41, 122, \ldots$ **(b)** $5, 8, 9, 8, 5, 0, \ldots$

(c) $2, 3, 1, 5, -3, 13, \ldots$ **(d)** $1, 6, 15, 28, 45, 66, \ldots$

(e) $5, 4, 7, -2, 25, -56, \ldots$

★ Challenge

An oncology team monitors a tissue mass over six weeks and records its volume mm^3as :$1, 3, 7, 13, 21, 31, \ldots$. find its general formula a_n. If volume exceeds 130 mm^3 by week 12, immediate surgery is required. Calculate a_{12} to determine if the patient needs surgery.

2-19
MATHEMATICS

Group Sequences

Chapter 2 Sequences

Point!

! For group sequences, consider the regularity of **the sequence before grouping** and the regularity of **the number of terms included in each group**.

Learning Outcomes

Solve various problems by dividing the sequence into groups to find a specific term or the sum of a particular group of terms.

Warm Up

Divide the sequence of natural numbers into groups as shown below. Solve the following problems.

2| 4, 6, 8 | 10, 12, 14, 16, 18 | 20, 22, …

1. Find the first term of the 10th group.
2. Find the sum of the terms included in the n-th group.
3. Find which group and which position 88 is in.

Explanation

Let $\{a_n\}$ be the sequence before grouping. Since it is an arithmetic sequence with the first term 2 and the common difference 2, $a_n = 2n$

Let $\{b_n\}$ be the sequence of the number of terms in each group. Since it is an arithmetic sequence with the first term 1 and the common difference 2, $b_n = 2n-1$

	the 1st group	the 2nd group	the 3rd group	
$\{a_n\}$	2	4, 6, 8	10, 12, 14, 16, 18	20, 22, …
$\{b_n\}$	1	3	5	

Sequence before grouping.

Sequence of term counts in each group.

1. Find the first term of the 10th group by finding what position the last term of the 9th group is in $\{a_n\}$.

   the 9th group the 10th group

   2 | 4, 6, 8 | 10, 12, 14, 16, 18 | … | … , ○|○, …

   one term three terms five terms

   The sum of the number of terms up to the 9th group is:

   $$\frac{1}{2}\times 9\times\{2\times 1+(9-1)\times 2\}=81$$

   Therefore, since the first term of the 10th group is the 82nd term of $\{a_n\}$:

   $2\times 82 = 164$

Consider the sum from the first term to the 9th term for $\{b_n\}$.
$S=\frac{1}{2}n\{2a+(n-1)d\}$

$a_n = 2n$

2 Find the sum using the ❶ last term, ❷ first term, and ❸ number of terms of the n–th group.

❶ Find the last term of the n-th group.

Finding what position the last term of the n–th group is in $\{a_n\}$:

$\frac{1}{2}n\{2\times1+(n-1)\times2\}=n^2\ldots\ldots$ (i)

Therefore, the last term of the n–th group is $2n^2$.

$a_n = 2n$

❷ Find the first term of the n-th group using the last term of the $(n-1)$-th group, similar to 1

the $(n-1)$-th group the n-th group

$\ldots, \circ \mid \circ, \ldots, \circ \mid \ldots$

From (i), the last term of the $(n-1)$-th group is the $(n-1)^2$-th term of $\{a_n\}$.

Therefore, the first term of the n–th group is the $\{(n-1)^2+1\}$–th term of $\{a_n\}$.

That is, the (n^2-2n+2)-th term, so:

$2(n^2-2n+2)$

❸ The number of terms in the n-th group is $2n-1$

$b_n = 2n-1$

Therefore, the required sum is the sum of an arithmetic sequence with the last term $2n^2$, the first term $2(n^2-2n+2)$, and number of terms $2n-1$:

$\frac{1}{2}(2n-1)\{2(n^2-2n+2)+2n^2\} = \underline{2(2n-1)(n^2-n+1)}$

3 First, find what position 88 is in $\{a_n\}$.

Solving $2n = 88$ gives $n=44$.

Here, from 2, the last terms of the $(n-1)$-th and n–th groups are the $(n-1)^2$–th and n^2–th terms of $\{a_n\}$ respectively.

Substitute specific numbers for n to check which group the 44th term belongs to.

Since $6^2 = 36$ and $7^2 = 49$, the 44th term is in the 7th group.

the 6th group the 7th group

..., (the 36th term) | ..., 88, ..., (the 49th term) |

(the 44th term)

Since $44-36=8$, 88 is **the 8th term of the 7th group**.

Try

Divide the sequence of natural numbers into groups as shown below. Solve the following problems.

1 | 3, 5 | 7, 9, 11 | 13, 15, 17, 19 | 21, 23, ...

1. Find the first term of the 10^{th} group.
2. Find the sum of the terms included in the 10^{th} group.
3. Find which group and which position 101 is in.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Divide the sequence of natural numbers into groups as shown below. Solve the following problems.

1. 1 | 2, 3 | 4, 5, 6 | 7, 8, 9, 10 | 11, 12, ...
   (a) Find the first term of the 10^{th} group.
   (b) Find the sum of the terms included in the 10^{th} group.
   (c) Find which group and which position 124 is in.

2. 1, 4 | 7, 10, 13, 16 | 19, 22, 25, 28, 31, 34 | 37, 40, ...
   (a) Find the first term of the 10^{th} group.
   (b) Find the sum of the terms included in the 10^{th} group.
   (c) Find which group and which position 100 is in.

★ Challenge

2, 6|10, 14, 18| 22, 26, 30, 34|......
(a) Find the first term of the 12th group.
(b) Derive a formula for the sum of all terms in the n-th group.
(c) Determine in which group and at which position the number 202 appears.
(d) Challenge extension: Find the total sum of all terms from Group 1 up to Group 12

Reflect

When analyzing an unfamiliar sequence of real-world data, what indicators or visual clues would lead you to choose difference sequence method over a group sequence method?

2-20

MATHEMATICS

Recurrence Relations (1)

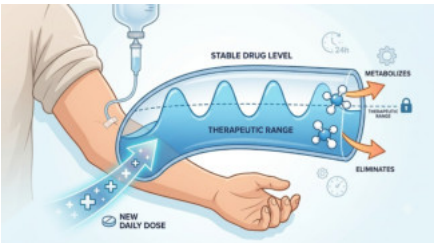
Chapter 2 Sequences

Learning Outcomes

Solve recurrence relations at their different levels in order to deduce the terms of the sequence.

Concept 5 · Recurrence and induction

When a doctor prescribes an antibiotic, the goal is to maintain a therapeutic amount of the drug in the patient's bloodstream. Every 24 hours, the patient takes a new dose, but their body also metabolizes and eliminates a fixed percentage of the drug already present.



To ensure the medication remains effective without reaching toxic levels, pharmacologists must model this daily loop. The amount of the drug tomorrow depends directly on the amount left over from today. This step-by-step dependency is the essence of a recurrence relation. To prove that this pattern remains safe over an infinite number of days, mathematicians use a logical domino effect called mathematical induction.

Guiding Question: How can we mathematically prove that a step-by-step, repeating process will remain stable and predictable forever?

Point!

! A sequence $\{a_n\}$ is determined, and $a_2,\ a_3,\ a_4, \dots$ can be found in order, if the following two conditions are given.

(a) The first term

(b) A relational expression to find a_{n+1} from a_n $(n = 1,\ 2,\ 3,\ \dots)$

The above (b) is called **a recurrence relation**. Hereafter, recurrence relations are assumed to hold for $n = 1,\ 2,\ 3,\ \dots$.

! Recurrence Relations Representing Arithmetic and Geometric Sequences.

$a_{n+1} = a_n + d$ **Arithmetic sequence with common difference d**

Adding d to a_n gives a_{n+1}

$a_{n+1} = ra_n$ **Geometric sequence with common ratio r**

Multiplying a_n by r gives a_{n+1}

Warm Up

Solve the following problems.

1 Find the 2nd through the 4th terms of the sequence $\{a_n\}$ determined by the following conditions.

$a_1 = 2, a_{n+1} = 3a_n + 2$

2 Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = -1, a_{n+1} = a_n + 5$ (b) $a_1 = 2, a_{n+1} = 3a_n$

Explanation

1 The 2nd. term is found by substituting $n = 1$ into the given recurrence relation.

$a_{1+1} = 3a_1 + 2$

$a_2 = 3 \times 2 + 2 \quad = 8$

The 3rd term is found using the 2nd term.

$a_{2+1} = 3a_2 + 2$

$a_3 = 3 \times 8 + 2$

$= 26$

Similarly, the 4th term is found using the 3rd term.

$a_{3+1} = 3a_3 + 2$

$a_4 = 3 \times 26 + 2$

$= 80$

Therefore, $\underline{a_2 = 8, a_3 = 26, a_4 = 80}$

2 (a) Since the sequence $\{a_n\}$ is an arithmetic sequence with the first term -1 and the common difference 5:

$a_n = -1 + (n-1) \times 5$

$a_n = a + (n-1)d$

$\underline{a_n = 5n - 6}$

(b) Since the sequence $\{a_n\}$ is a geometric sequence with the first term 2 and the common ratio 3:

$\underline{a_n = 2 \times 3^{n-1}}$

$a_n = ar^{n-1}$

Try

Solve the following problems.

1 Find the 2nd. through the 4th terms of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = -6,\ a_{n+1} = 2a_n + 3$ (b) $a_1 = 2,\ a_{n+1} = 3a_n - n$

2 Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 1,\ a_{n+1} = a_n + 3$ (b) $a_1 = 4,\ a_{n+1} = a_n - 2$ (c) $a_1 = 3,\ a_{n+1} = 5a_n$

(d) $a_1 = -3,\ a_{n+1} = -2a_n$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the 2nd. through the 4th. terms of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 1,\ a_{n+1} = -2a_n + 5$

(b) $a_1 = -6,\ a_{n+1} = a_n + n^2 - 2n$

(c) $a_1 = 3,\ a_{n+1} = 3a_n - n$

(d) $a_1 = -2,\ a_{n+1} = a_n - n$

(e) $a_1 = 2,\ a_{n+1} = a_n + 2n$

(f) $a_1 = -2,\ a_{n+1} = a_n + 3^n$

2 Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 1,\ a_{n+1} = a_n + 6$ (b) $a_1 = 3,\ a_{n+1} = a_n - 5$ (c) $a_1 = 4,\ a_{n+1} = 3a_n$

(d) $a_1 = 2,\ a_{n+1} = -5a_n$

(e) $a_1 = -2,\ a_{n+1} = a_n + 3$ (f) $a_1 = 4,\ a_{n+1} = a_n - 4$ (g) $a_1 = 1,\ a_{n+1} = 4a_n$

(h) $a_1 = -2,\ a_{n+1} = -4a_n$

★ Challenge

Solve the recurrence relation for a_n : $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ $for\ n \geq 3$
Given initial conditions: $a_0 = 0$, $a_1 = 1$, and $a_2 = 6$

2-21

MATHEMATICS

Recurrence Relations (2)

Chapter 2 Sequences

Point!

! Recurrence relations where the difference sequence is known

$a_{n+1} = a_n + f(n)$ $f(n)$ is the general term of the difference sequence of $\{a_n\}$.

When $n \geq 2$, $a_n = a_1 + \sum_{k=1}^{n-1} f(k)$

Learning Outcomes

Solve recurrence relations at their different levels in order to deduce the terms of the sequence.

An expression in terms of n

Can be transformed to $a_{n+1} - a_n = f(n)$

Warm Up

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

$a_1 = 1, a_{n+1} = a_n + 2n - 3$

Explanation

Since $a_{n+1} - a_n = 2n - 3$, the general term of the difference sequence of $\{a_n\}$ is $2n - 3$

Therefore, when $n \geq 2$:

$a_n = 1 + \sum_{k=1}^{n-1} (2k - 3)$

$= 1 + 2 \times \frac{1}{2}(n-1)\{(n-1) + 1\} - 3(n-1)$

$= n^2 - 4n + 4 = (n-2)^2$

This formula also holds when $n = 1$

Therefore, $a_n = (n-2)^2$

Try

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 2, a_{n+1} = a_n + n^2$ **(b)** $a_1 = 1, a_{n+1} = a_n + 3^n$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 2,\ a_{n+1} = a_n + 2n$

(b) $a_1 = 2,\ a_{n+1} = a_n + 3n^2$

(c) $a_1 = 1,\ a_{n+1} = a_n + 2^n$

(d) $a_1 = 2,\ a_{n+1} = a_n + 3n + 1$

(e) $a_1 = 2,\ a_{n+1} = a_n + 2n^2 + n$

(f) $a_1 = 2,\ a_{n+1} = a_n - 3^n$

(g) $a_1 = 4,\ a_{n+1} = a_n - 4^n$

★ Challenge

A tech startup launches with a 400,000 LE cash reserve ($a_1 = 4$ hundred thousand). They burn an amount of 4n thousand LE at the end of each quarter:
$a_1 = 4,\ a_{n+1} = a_n - 0.04n$
Find an explicit formula for the general term a_n to predict the company's cash reserve at the start of any quarter.

2-22

MATHEMATICS

Recurrence Relations (3)

Chapter 2 Sequences

Learning Outcomes

Solve recurrence relations at their different levels in order to deduce the terms of the sequence.

Point!

! Steps to find the general term from a recurrence relation of the form $a_{n+1} = pa_n + q$.

1. Replace both a_{n+1} and a_n in the recurrence relation with c.
2. Solve the equation from 1.
3. Substitute the solution from 2 into $\mathbf{a_{n+1} - c = p(a_n - c)}$.
4. Letting $a_n - c = b_n$, $\{b_n\}$ is a geometric sequence with common ratio p.
5. Find the general term of $\{a_n\}$ using the general term of $\{b_n\}$.

Warm Up

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

$a_1 = 1, a_{n+1} = 3a_n + 4$

Explanation

Replacing both a_{n+1} and a_n in the given recurrence relation $a_{n+1} = 3a_n + 4$ with c :

$c = 3c + 4$

Solving this gives $c = -2$

Therefore, $a_{n+1} + 2 = 3(a_n + 2)$. ...... (i)

Substitute $c = -2$ into $a_{n+1} - c = p(a_n - c)$.

Here, letting $a_n + 2 = b_n$ ......(ii), we can express it as $a_{n+1} + 2 = b_{n+1}$

Substituting these into (i) gives $b_{n+1} = 3b_n$, so $\{b_n\}$ is a geometric sequence with common ratio 3

Also, the first term b_1 of $\{b_n\}$ is, from (ii), $b_1 = a_1 + 2$

$= 1 + 2 \quad = 3$

Therefore, $b_n = 3 \times 3^{n-1}$

$\{b_n\}$ is a geometric sequence with the first term 3, the common ratio 3

$= 3^n$

Substituting this into (ii) gives $a_n + 2 = 3^n$

$\mathbf{a_n = 3^n - 2}$

Try

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 1,\ a_{n+1} = 5a_n + 8$ (b) $a_1 = 4,\ a_{n+1} = -2a_n + 3$

Exercise

Find the general term of the sequence $\{a_n\}$ determined by the following conditions.

(a) $a_1 = 2,\ a_{n+1} = 2a_n - 1$ **(b)** $a_1 = -3,\ a_{n+1} = 4a_n + 3$

(c) $a_1 = 3,\ a_{n+1} = 3a_n - 2$ **(d)** $a_1 = -2,\ a_{n+1} = -3a_n + 4$

(e) $a_1 = 1,\ a_{n+1} = \frac{1}{3}a_n + 2$ **(f)** $a_1 = 2,\ a_{n+1} = \frac{2}{3}a_n - \frac{1}{2}$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

2-23

MATHEMATICS

Mathematical Induction for Proving Equalities

Chapter 2 Sequences

Point!

! Proof by mathematical induction

Procedure to prove that “Condition (A) holds for all natural numbers n”

[1] Show that (A) holds when $\boldsymbol{n = 1}$

[2] Assume that (A) holds when $\boldsymbol{n = k}$, and write the equation that holds then.

By transforming that equation, show that (A) also holds when $\boldsymbol{n = k + 1}$

Learning Outcomes

Apply the principle of mathematical induction to prove the validity of mathematical equalities and identities.

Warm Up

Use mathematical induction to prove that $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$

Explanation

Let the given equality be (A).

[1] When $n = 1$:

$$\text{LHS} = 1,\ \text{RHS} = \frac{1}{2} \times 1 \times (1 + 1) = 1$$

Therefore, (A) holds when $n = 1$

[2] Assume (A) holds when $n = k$:

$$1 + 2 + 3 + \ldots + k = \frac{1}{2}k(k + 1) \ldots\ldots \text{(i)}$$

When $n = k + 1$, from (i):

$$\text{LHS} = \underline{1 + 2 + 3 + \ldots + k} + (k+1)$$

Derive the expression that holds for $n = k + 1$ using (i).

$$= \underline{\frac{1}{2}k}(k + 1) + (k + 1)$$

Do not expand; factor out $\frac{1}{2}(k + 1)$.

$$= \frac{1}{2}(k + 1)(k + 2)$$

$$= \frac{1}{2}(k + 1)\{(k + 1) + 1\}$$

$$\text{RHS} = \frac{1}{2}(k + 1)\{(k + 1) + 1\}$$

Transform into the form obtained by substituting $n = k + 1$ into the RHS of (A).

Therefore, (A) also holds when $n = k + 1$

From [1] and [2], (A) holds for all natural numbers.

Try

Use mathematical induction to prove the following equalities.

(a) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

(b) $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Use mathematical induction to prove the following equalities.

(a) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$

(b) $3 + 3 \times 4 + 3 \times 4^2 + \ldots + 3 \times 4^{n-1} = 4^n - 1$

(c) $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

(d) $1 + 10 + 10^2 + \ldots + 10^n = \frac{1}{9}\left(10^{n+1} - 1\right)$

2-24
MATHEMATICS

Mathematical Induction for Proving Inequalities

Chapter 2 Sequences

Point!

! When proving equalities or inequalities involving a^n, multiply both sides of the assumed expression by a

Learning Outcomes

Apply the principle of mathematical induction to prove the validity of algebraic inequalities.

Warm Up

When n is a natural number greater than or equal to 4, prove the following inequality.

$2^n > 3n$

Explanation

Let the given inequality be (A).

[1] When $n = 4$, LHS $= 2^4 = 16$, RHS $= 3 \times 4 = 12$

Therefore, (A) holds when $n = 4$

[2] Assuming $k \geq 4$ and (A) holds when $n = k$,

$2^k > 3k$

Multiplying both sides by 2 gives

$2^{k+1} > 6k$ ...... (i)

Here, comparing $6k$ and $3(k+1)$:

When $k \geq 4$, $6k - 3(k+1) = 6k - 3k - 3$

$= 3k - 3 > 0$

Therefore, $6k > 3(k+1)$ ......(ii)

From (i) and (ii), since $2^{k+1} > 6k > 3(k+1)$,

$2^{k+1} > 3(k+1)$

This shows that (A) also holds when $n = k + 1$

From [1] and [2], (A) holds for all natural numbers greater than or equal to 4

Since we are proving this for $n \geq 4$ more, show it holds when $n = 4$

Using this expression, show (A) for $n = k + 1$, which is $2^{k+1} > 3(k+1)$.

Since the RHS when substituting $n = k + 1$ into (A) is $3(k+1)$, show $2^{k+1} > 6k > 3(k+1)$.

Try

When n is a natural number greater than or equal to 3, prove the following inequality.

$3^n > 5n+1$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1. When n is a natural number, prove the following inequality.

   $3^n > 2n$

2. When n is a natural number greater than or equal to 4, prove the following inequality.

   $2^n > 3n+1$

★ Challenge

When $h > 0$ and n is a natural number greater than or equal to 3, prove the following inequality.

$(1+h)^n > 1 + nh^2$

2-25
MATHEMATICS

Proof of Being a Multiple by Mathematical Induction

Chapter 2 Sequences

Point!

! When proving that something is a multiple of a, use the assumed expression to show it takes the form $a \times$ (integer).

Learning Outcomes

Apply the principle of mathematical induction to prove the validity of divisibility.

Warm Up

Prove that $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9 for all natural numbers n, using mathematical induction.

Explanation

Let "(A)" be "$n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9".

[1] When $n = 1$,

$$1^3+(1+1)^3+(1+2)^3 = 1+8+27$$

$$= 36$$

36 is a multiple of 9

Therefore, (A) holds when $n = 1$.

[2] Assuming (A) holds when $n = k$, it can be expressed using an integer m as:

$$k^3 + (k+1)^3 + (k+2)^3 = 9m$$

Using this equation, show that (A) holds when $n = k + 1$

Considering when $n = k+1$:

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9m - k^3 + (k+3)^3$$

Since $(k+1)^3 + (k+2)^3 = 9m - k^3$ from $k^3 + (k+1)^3 + (k+2)^3 = 9m$: $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9m - k^3 + (k+3)^3$ $n = k+1$

$$= 9m - k^3 + k^3 + 9k^2 + 27k + 27 \quad = 9\left(m + k^2 + 3k + 3\right)$$

Since $m + k^2 + 3k + 3$ is an integer, $9(m + k^2 + 3k + 3)$ is a multiple of 9

Therefore, $(k+1)^3 + (k+2)^3 + (k+3)^3$ is a multiple of 9, and (A) also holds when $n = k+1$.

From [1] and [2], (A) holds for all natural numbers.

Try

Prove that $n^3 + 2n$ is a multiple of 3 for all natural numbers n, using mathematical induction.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Prove that $2n^3 - 3n^2 + n$ is a multiple of 6 for all natural numbers n, using mathematical induction.

2 Prove that $4n^3 - n$ is divisible by 3 for all natural numbers n, using mathematical induction.

★ Challenge

Prove that for any integer $n \geq 1$:
$(n+1)^n - 1$ is a multiple of n^2.

2-26
MATHEMATICS

Recurrence Relations and Mathematical Induction

Chapter 2 Sequences

Warm Up

For the sequence $\{a_n\}$ determined by the conditions $a_1 = 3,\ a_{n+1} = \dfrac{a_n^2 - 1}{n+1}$, solve the following problems.

1 Find a_2, a_3, a_4 and estimate the general term of this sequence.

2 Prove that the estimation in 1 is correct using mathematical induction.

Explanation

1 From the given recurrence relation:

$a_2 = \dfrac{a_1^2 - 1}{1+1} = \dfrac{3^2 - 1}{2} = \dfrac{8}{2} = 4$ Similarly, $a_3 = 5,\ a_4 = 6$

Therefore, we can estimate $a_n = n + 2$

2 Let $a_n = n + 2$ be (A).

[1] When $n = 1$, substituting n=1 into (A) gives:

$a_1 = 1 + 2 = 3$

Therefore, (A) holds when $n = 1$

[2] Assuming (A) holds when $n = k$,

$a_k = k + 2$

At this time, finding a_{k+1} from the given recurrence relation:

$a_{k+1} = \dfrac{{a_k}^2 - 1}{k+1}$

$= \dfrac{(k+2)^2 - 1}{k+1}$

$= k + 3$

$= (k+1) + 2$

This shows that (A) also holds when $n = k + 1$

From [1] and [2], (A) holds for all natural numbers.

Therefore, the estimation in 1 is correct.

Learning Outcomes

Employ mathematical induction to prove and validate recurrence relations of sequences.

$\{a_n\},\ 3,\ 4,\ 5,\ 6,\ \ldots$

Substitute $a_k = k + 2$

Transform into the form obtained by substituting $n = k + 1$ into the RHS of (A)

Try

For the sequence $\{a_n\}$ determined by $a_1 = -1$, $a_{n+1} = a_n^2 + 2na_n - 2$, solve the following problems.

1. Find a_2, a_3, a_4 and estimate the general term.
2. Prove that the estimation in 1 is correct using mathematical induction.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1. For the sequence $\{a_n\}$ determined by $a_1 = 1$, $a_{n+1} = a_n^2 - n^2 a_n + (n+1)^2$, solve the following problems.

   (a) Find a_2, a_3, a_4 and estimate the general term.

   (b) Prove that the estimation in (a) is correct using mathematical induction.

2. For the sequence $\{a_n\}$ determined by $a_1 = 1$, $a_{n+1} = \frac{a_n - 4}{a_n - 3}$, solve the following problems.

   (a) Find a_2, a_3, a_4 and estimate the general term.

   (b) Prove that the estimation in (a) is correct using mathematical induction.

★ Challenge

Three numbers, A, B, C in that order, are in geometric sequence with common ratio r. Given further that A, 2B, C in that order are in arithmetic sequence, determine the possible values of r.

Reflect

How can mathematical induction help us confirm the correctness of a general formula for a sequence defined by a recurrence relation? Explain your answer.

3-1

MATHEMATICS

Complex Numbers

Chapter 3 Complex Numbers and Equations

Concept 1 · Complex number

When a medical technician monitors a patient's heart using an electrocardiogram (ECG), the screen displays a rhythmic, wave-like pattern. Similarly, an MRI machine uses powerful magnetic fields and radio waves to construct detailed images of organs inside the human body.

Learning Outcomes

Identify the complex number ($z = a + bi$) and its components (the real part and the imaginary part).

These biological signals are fundamentally waves, and tracking them requires tracking both their size (amplitude) and their timing (phase shift). While standard numbers fail to capture both pieces of data simultaneously, a unique mathematical framework solves this problem perfectly by treating numbers as coordinates on a plane.

Guiding Question: How can complex numbers help us expand our understanding of mathematics so that we can solve problems that seem impossible using only real numbers?

Point!

❗ The number whose square is -1 is denoted by i. That is, $i^2 = \underline{-1}$

❗ A number expressed in the form $a + bi$ using two real numbers $a,\ b$ is called a **complex number**.

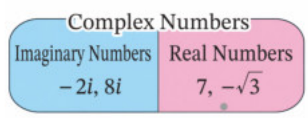
The part a is called the **Real Part**, and the part b is called the **Imaginary Part**.

When $b = 0$, $a + bi$ becomes a real number.

When $b \neq 0$, $a + bi$ is called an **imaginary number**.

❗ Equality of Complex Numbers

- When $a,\ b,\ c,\ d$ are real numbers, if $a + bi = c + di$, then $\underline{a} = \underline{c},\ \underline{b} = \underline{d}$
- When $a,\ b$ are real numbers, if $a + bi = 0$, then $\underline{a} = \underline{0},\ \underline{b} = \underline{0}$



Real numbers are also part of Complex numbers.

i is not included in the imaginary part.

The real and imaginary parts of both sides are equal.

$0 = 0 + 0i$

Warm Up

Solve the following problems.

1 State the real and imaginary parts of the following complex numbers.

(a) $3-2i$ **(b)** $\dfrac{4-\sqrt{5}i}{3}$ **(c)** -8 **(d)** $2i$

2 Find the values of real numbers $x,\ y$ that satisfy the following equation.

$(2+i)x+(1+i)y=2+3i$

Explanation

1 **(a)** $\underbrace{3}_{a}-\underbrace{2}_{b}i$ **Real Part: 3, Imaginary Part: −2**

i is not included in the imaginary part.

(b) $\dfrac{4-\sqrt{5}i}{3}=\dfrac{4}{3}-\dfrac{\sqrt{5}}{3}i$ **Real Part: $\frac{4}{3}$, Imaginary Part: $-\frac{\sqrt{5}}{3}$**

Express it in the form $a+bi$ first.

(c) $-8=-8+0i$ **Real Part: −8, Imaginary Part: 0**

(d) $2i=0+2i$ **Real Part: 0, Imaginary Part: 2**

2 $(2+i)x+(1+i)y=2+3i$

Expand the LHS and rearrange into the form $a+bi$.

$2x+ix+y+iy=2+3i$

$(\underbrace{2x+y}_{a})+(\underbrace{x+y}_{b})i=\underbrace{2}_{a}+\underbrace{3}_{b}i$

Compare the real and imaginary parts of both sides.

Therefore,

$$\begin{cases} 2x+y=2 \\ x+y=3 \end{cases}$$

Solving this system of equations gives $x=-1,\ y=4$

Try

Solve the following problems.

1 State the real and imaginary parts of the following complex numbers.

(a) $-3+5i$ **(b)** $\dfrac{-1-\sqrt{3}i}{2}$ **(c)** 1 **(d)** $3i$

2 Find the values of real numbers $x,\ y$ that satisfy the following equations.

(a) $(x-6)+(x+3y)i=0$ **(b)** $(3+2i)x+(4-i)y=6-7i$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 State the real and imaginary parts of the following complex numbers.

(a) $2 + 5i$ **(b)** $-25 + 13i$ **(c)** $\dfrac{2 - \sqrt{5}i}{3}$ **(d)** $\dfrac{7 - \sqrt{2}i}{2}$

(e) -4 **(f)** $\dfrac{1}{4}$ **(g)** $\sqrt{3}i$ **(h)** $-\mathrm{i}$

2 Find the values of real numbers $x,\ y$ that satisfy the following equations.

(a) $(2x + y) + (3x - y)i = 8 + 7i$ **(b)** $(x + 1) - 5i = 4 + (2y + 1)i$

(c) $(x - 2) + (x + y)i = 0$ **(d)** $(3 + 2i)x + (1 - i)y = 7 + 3i$

(e) $(1 - i)(x + yi) = 2 + i$

★ Challenge

A sound engineer is programming a stereo echo plugin where complex numbers track the volume and timing of audio waves. To perfectly center the sound between the left speaker (volume multiplier x) and right speaker (volume multiplier y), the two audio paths must balance according to this equation: $(1 + 2i)(x + i) = x(y + i)$ Find the exact values of the real numbers x and y.

3-2

MATHEMATICS

Operations with Complex Numbers (1)

Chapter 3 Complex Numbers and Equations

Point!

! Calculations involving complex numbers can be performed in the same way as algebraic expressions. However, if i^2 appears, substitute $\underline{i^2} = \underline{-1}$

! Powers of i should be calculated by expressing them in terms of i^2

Example: $i^8 = \underline{(i^2)^4} = (-1)^4 = 1$ $\quad i^7 = \underline{i \times (i^2)^3} = i \times (-1)^3 = -i$

Learning Outcomes

Identify the complex number ($z = a + bi$) and its components (the real part and the imaginary part).
Perform arithmetic operations (addition, subtraction, multiplication, division) on complex numbers and put them in their simplest form.

When the exponent is odd, express it using i and i^2.

Warm Up

Calculate the following.

(a) $(3-2i)-(2+5i)$ **(b)** $(i-2)^2$ **(c)** $i^6 + i^{27} + i^{17}$

Explanation

(a) $(3-2i)-(2+5i)$

$= 3-2i-2-5i$

$= 1-7i$

Calculate treating i as a variable.

Answer in the form $a + bi$

(b) $(i-2)^2$

$= i^2 - 4i + 4$

$= -1 - 4i + 4$

$= 3 - 4i$

Substitute $i^2 = -1$

(c) $i^6 + i^{27} + i^{17}$

$= (i^2)^3 + i \times (i^2)^{13} + i \times (i^2)^8$

$= (-1)^3 + i \times (-1)^{13} + i \times (-1)^8$

$= -1 - i + i$

$= -1$

Express each power of i using i^2

Misconceptions

Mixing real and imaginary parts: Adding or subtracting unlike terms together instead of keeping real and imaginary parts separate.
Errors in multiplication: When expanding brackets, forgetting that $i \times i = i^2 = -1$

Try

Calculate the following.

(a) $(4+6i)+(-3+2i)$ **(b)** $(2+3i)(1-2i)$ **(c)** $i + i^2 + i^3 + i^4$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Calculate the following.

(a) $(7+4i)-(3+5i)$

(b) $(-4+3i)+(5-6i)$

(c) $(3-3i)-(5-3i)$

(d) $\left(\sqrt{2}+3i\right)\left(\sqrt{2}+i\right)$

(e) $(2+3i)+(5+8i)$

(f) $\left(\frac{2}{5}-\frac{1}{2}i\right)-\left(\frac{1}{3}+\frac{3}{7}i\right)$

(g) $(2i-1)^2$

(h) $\left(\frac{2}{i}+i\right)\left(\frac{2}{i}-i\right)$

(i) $(2+i)^3+(2-i)^3$

(j) i^5+i^3+i

(k) $i^9+i^{14}+i^{11}$

(l) $i^{12}+i^{18}+i^{15}$

★ Challenge

Given that x is a real number, solve the equation:

$$(7x-7i)\left(x^2+xi\right)=14x^2i^{40}$$

3-3
MATHEMATICS

Operations with Complex Numbers (2)

Chapter 3 Complex Numbers and Equations

Point!

! Complex numbers $a + bi$ and a − bi are called **Complex Conjugates** .

! The product of conjugate complex numbers becomes a real number.

Example: $(2+3i)(2-3i) = 4-9i^2$

$= 4-9\times(-1) \qquad = 4+9 \qquad = 13$

! For fractional expressions containing imaginary numbers in the denominator, first multiply both the numerator and denominator by **the complex conjugate of the denominator** to make the denominator a real number.

Example: $\dfrac{1-i}{2+3i} = \dfrac{(1-i)(2-3i)}{(2+3i)(2-3i)}$

$= \dfrac{-1-5i}{13}$

Learning Outcomes

Identify the complex number (z = a + bi) and its components (the real part and the imaginary part).

Perform arithmetic operations (addition, subtraction, multiplication, division) on complex numbers and put them in their simplest form.

The signs of their imaginary parts are opposite.

Warm Up

Solve the following problems.

1 State the complex conjugate of the following complex numbers.

(a) $2+3i$ **(b)** $7-i\sqrt{11}$ **(c)** 6 **(d)** $-5i$

2 Calculate the following and answer in the form $a + bi$.

(a) $\dfrac{2+i}{2-i}$ **(b)** $\dfrac{3-i}{2i}$

Explanation

1 Reverse the sign of the imaginary part.

(a) $2-3i$ **(b)** $7+i\sqrt{11}$ **(c)** 6 **(d)** $5i$

Conjugate of $6+0i$ is $6-0i$

Conjugate of $0-5i$ is $0+5i$

2 **(a)** $\dfrac{2+i}{2-i}$

Multiply numerator and denominator by $2+i$

$= \dfrac{(2+i)^2}{(2-i)(2+i)} \qquad = \dfrac{4+4i+i^2}{4-i^2}$

$= \dfrac{4+4i+(-1)}{4-(-1)} \qquad = \dfrac{3+4i}{5}$

$= \dfrac{3}{5} + \dfrac{4}{5}i$

Answer in the form $a + bi$

(b) $\frac{3-i}{2i}$

Multiply numerator and denominator by $-2i$

$$=\frac{(3-i)(-2i)}{2i(-2i)} \qquad =\frac{-6i+2i^2}{-4i^2}$$

$$=\frac{-6i+2\times(-1)}{-4\times(-1)}$$

$$=\frac{-{}^{1}\cancel{2}-{}^{3}\cancel{6}i}{\cancel{4}_{2}}$$

Simplify the fraction.

$$=-\frac{1}{2}-\frac{3}{2}i$$

Answer in the form $a+bi$.

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Solve the following problems.

1 State the complex conjugate of the following complex numbers.

(a) $1-4i$ **(b)** $\sqrt{2}+i\sqrt{10}$

(c) 2 **(d)** $-8i$

2 Calculate the following and answer in the form $a+bi$.

(a) $\frac{3+i}{1+2i}$ **(b)** $\frac{1-i}{i}$

Exercise

Solve the following problems.

1 State the complex conjugate of the following complex numbers.

(a) $3+4i$ **(b)** $9+8i$ **(c)** $8+i\sqrt{3}$

(d) 5 **(e)** -15 **(f)** $\sqrt{7}$

(g) i **(h)** $-50i$ **(i)** $i\sqrt{6}$

2 Calculate the following and answer in the form $a+bi$.

(a) $\frac{1+i}{1-i}$ **(b)** $\frac{3+i}{2+i}$ **(c)** $\frac{\sqrt{5}+i}{\sqrt{5}-i}$ **(d)** $\frac{5i}{1+2i}$

(e) $\frac{5i}{4+3i}$ **(f)** $\frac{1-3i}{3-2i}$ **(g)** $\frac{1}{i}$ **(h)** $\frac{2+i}{3i}$

★ Challenge

Perform the indicated operation and write your answer in standard form.

$$\left(\frac{i(10-12i)}{(2+i)(-1+4i)}\right)^{2027}$$

3-4

MATHEMATICS

Square Roots of Negative Numbers

Chapter 3 Complex Numbers and Equations

Point!

! When $a > 0$, $\sqrt{-a} = i\sqrt{a}$

! When the inside of the square root is a negative number, calculate after expressing it in a form in terms of i.

Learning Outcomes

Calculate square roots of negative numbers and complex numbers .

Warm Up

Solve the following problems.

1 Express the square roots of −6 in terms of i.

2 Calculate the following.

(a) $\sqrt{-12} \times \sqrt{-6}$ **(b)** $\dfrac{\sqrt{8}}{\sqrt{-18}}$

Explanation

1 The square roots of −6 are $\pm\sqrt{-6} = \pm i\sqrt{6}$

2 **(a)** $\sqrt{-12} \times \sqrt{-6}$

First express in terms of i

$= i\sqrt{12} \times i\sqrt{6}$

$= 2i\sqrt{3} \times i\sqrt{6}$

$= 6i^2\sqrt{2}$

Substitute $i^2 = -1$

$= 6\sqrt{2} \times (-1) = -6\sqrt{2}$

(b) $\dfrac{\sqrt{8}}{\sqrt{-18}}$

$= \dfrac{2\cancel{\sqrt{2}}}{3\cancel{\sqrt{2}}\, i}$

Multiply numerator and denominator by $-3i$ (or simpler, just i).

$= \dfrac{-6i}{9}$

$= -\dfrac{2}{3}i$

Misconceptions

Thinking $\sqrt{-n}$ is undefined or impossible: Students may believe negative numbers cannot have square roots at all, rather than understanding that complex numbers extend the system to make this possible.

Confusing signs: Some assume $\sqrt{-n} = -\sqrt{n}$, which is incorrect. The correct form is $\sqrt{-n} = i\sqrt{n}$.

Try

Solve the following problems.

1 Express the following numbers in terms of i.

(a) $\sqrt{-7}$ **(b)** $\sqrt{-27}$ **(c)** $-\sqrt{-12}$ **(d)** Square roots of -12

2 Calculate the following.

(a) $\sqrt{-3} \times \sqrt{-6}$ **(b)** $\dfrac{\sqrt{27}}{\sqrt{-3}}$ **(c)** $(3 + \sqrt{-2})^2$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Express the following numbers in terms of i.

(a) $\sqrt{-2}$ **(b)** $\sqrt{-9}$

(c) $\sqrt{-45}$ **(d)** $-\sqrt{-24}$

(e) $-\sqrt{-36}$ **(f)** Square roots of -2

(g) Square roots of -18 **(h)** Square roots of -16

2 Calculate the following.

(a) $\sqrt{-2} \times \sqrt{-6}$ **(b)** $\sqrt{-28} \times \sqrt{-35}$

(c) $\dfrac{\sqrt{8}}{\sqrt{-2}}$ **(d)** $\dfrac{\sqrt{27}}{\sqrt{-12}}$

(e) $\sqrt{-32} + \sqrt{-98} + \sqrt{-8}$ **(f)** $\sqrt{-98} - \sqrt{-72} + \sqrt{-50}$

(g) $(\sqrt{3} + \sqrt{-5})^2$ **(h)** $(2 + \sqrt{-3})(2 - \sqrt{-3})$

★ Challenge

Solve: $(2x - 8)^2 = -10$

Reflect

Looking back at how expanding our number system from standard real lines to two-dimensional planes allows us to calculate things like tissue impedance and audio processing, how does this change your view of what makes a mathematical concept "real" versus "imaginary"?

3-5

MATHEMATICS

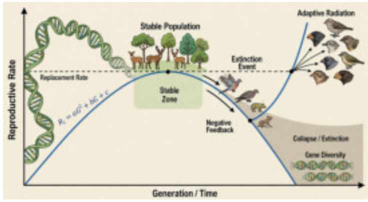
Solutions of Quadratic Equations

Chapter 3 Complex Numbers and Equations

Learning Outcomes

Solve quadratic equations in the set of complex numbers.

Concept 2 · Quadratic equations



Evolutionary biologists study how genetic traits spread through populations over time, determining whether a species adapts, stabilizes, or faces extinction. To model interactions between competing genetic variations under natural selection, they use mathematics to represent reproductive rates and generational effects. At the balance point between genetic traits and environmental pressures, the equations form quadratic expressions, which help predict whether a population is stable or vulnerable to collapse.

Guiding Question: "How can quadratic equations help us determine whether a system in life science or engineering will remain stable or collapse?"

Point!

! When the inside of the square root is negative, express it in a **form** in terms of $\boldsymbol{i}$

! How to solve Quadratic Equations.

(i) Solve using the concept of Square Roots.

The solutions of $x^2 = a$ are $x = \pm\sqrt{a}$

For $(x+m)^2 = a$, transform to $x + m = \pm\sqrt{a}$, so $x = -m \pm \sqrt{a}$

(ii) Solve using Factorisation.

The solutions of $x(x-a)=0$ are $x=$ **0 or a**.

The solutions of $(x-a)(x-b)=0$ are $x=$ **a or b**

The solution of $(x-a)^2=0$ is $x=a$

The solutions of $(ax-b)(cx-d)=0$ are $x=\dfrac{b}{a}$ or $\dfrac{d}{c}$

(iii) Solve using the Quadratic Formula.

The solutions of $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Warm Up

Solve the following quadratic equations.

(a) $x^2 + 8 = 0$ **(b)** $2x^2 - 7x - 9 = 0$ **(c)** $2x^2 + 2\sqrt{6}x + 11 = 0$

Explanation

(a) $x^2 + 8 = 0$ — Create the form "$x^2 =$ ".

$x^2 = -8$ — Remove the square on the LHS and add $\pm\sqrt{\ }$ to the RHS.

$x = \pm\sqrt{-8}$ — Express in terms of i.

$x = \pm i\sqrt{8}$

$x = \pm 2i\sqrt{2}$

(b) $2x^2 - 7x - 9 = 0$ — Use the cross-multiplication method.

$(x + 1)(2x - 9) = 0$

$x = -1 \text{ or } \frac{9}{2}$

(c) $2x^2 + 2\sqrt{6}x + 11 = 0$ — Use the Quadratic Formula.

$$x = \frac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4 \times 2 \times 11}}{2 \times 2}$$

$$x = \frac{-2\sqrt{6} \pm \sqrt{-64}}{4}$$ — Express in terms of i.

$$x = \frac{-2\sqrt{6} \pm i\sqrt{64}}{4}$$

$$x = \frac{-2\sqrt{6} \pm 8i}{4}$$ — Simplify the fraction.

$$x = \frac{-\sqrt{6} \pm 4i}{2}$$

Try

Solve the following quadratic equations.

(a) $x^2 + 4 = 0$ **(b)** $(x - 3)^2 = -8$

(c) $2x^2 + 5x - 3 = 0$ **(d)** $2x^2 + 2\sqrt{3}x + 5 = 0$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Solve the following quadratic equations.

(a) $x^2 + 49 = 0$ **(b)** $9x^2 - 4 = 0$

(c) $(x-2)^2 = 5$ **(d)** $(x+6)^2 = -6$

(e) $2x^2 + 7x = 0$ **(f)** $2x^2 + 5x - 12 = 0$

(g) $x^2 + x + 1 = 0$ **(h)** $3x^2 - 5x + 3 = 0$

(i) $x^2 - 2\sqrt{3}x + 5 = 0$ **(j)** $3x^2 + 2\sqrt{2}x + 1 = 0$

2 Solve the following quadratic equations.

(a) $x^2 - x + 2 = 0$ **(b)** $(x+4)^2 = -18$

(c) $x^2 - 12 = 0$ **(d)** $3x^2 - 4x - 1 = 0$

(e) $x^2 - 8x + 15 = 0$ **(f)** $4x^2 + 3 = 0$

(g) $3x^2 - \sqrt{5}x + 1 = 0$ **(h)** $6x^2 - 12x + 15 = 0$

(i) $(2x-3)^2 = 7$

★ Challenge

A research team studies the vibrations of a thin membrane in a medical device that uses ultrasound waves. The relationship between the vibration amplitude x and the rate of energy transfer R is given by:

$$R = 3x^2 + 2\sqrt{3}x + 2$$

Determine the values of x that make the system reach equilibrium (when the energy transfer rate is zero). And explain the meaning of the result.

3-6

MATHEMATICS

Discriminant of Quadratic Equations

Chapter 3 Complex Numbers and Equations

Learning Outcomes

Determine the nature of its roots using the discriminant and use them to solve math problems.

Point!

! The expression used to determine the type and number of solutions of the quadratic equation $ax^2 + bx + c = 0$ is called the Discriminant, denoted by D.

$D =$ **$b^2 - 4ac$**

Inside the square root of the Quadratic Formula.

!

When $D > 0$, it has **two distinct real solutions**.

When $D = 0$, it has **a repeated solution (one real solution)**.

When $D \geq 0$, **it has real solutions**

When $D < 0$, it has **two distinct imaginary solutions**.

Warm Up

Solve the following problems.

1 Determine the nature of the solutions for the quadratic equation $3x^2 + 4x + 7 = 0$

2 Find the range of values for the constant m for which the quadratic equation $x^2 - mx + m^2 - 3m - 9 = 0$ has real solutions.

3 Let k be a real constant. Determine the nature of the solutions for the quadratic equation $x^2 + kx + 3 - k = 0$

Explanation

1 Let D be the discriminant of $3x^2 + 4x + 7 = 0$

Substitute $a = 3, b = 4, c = 7$ into the discriminant.

$D = 4^2 - 4 \times 3 \times 7$ $\therefore$ $D = 16 - 84 = -68 < 0$

Therefore, it has two distinct imaginary solutions.

Misconceptions

Misinterpreting the result

D < 0: thinking there are "no solutions at all," while the correct answer is two complex solutions.

D = 0: thinking there are "no solutions," while the correct answer is one repeated real solution.

D > 0: thinking there is "only one solution," while the correct answer is two distinct real solutions

2 Let D be the discriminant of $x^2 - mx + m^2 - 3m - 9 = 0$ Since $D \geq 0$,

$(-m)^2 - 4\left(m^2 - 3m - 9\right) \geq 0$

$-3m^2 + 12m + 36 \geq 0$

$3m^2 - 12m - 36 \leq 0$

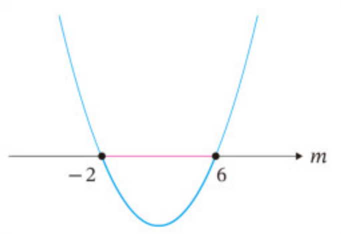
$m^2 - 4m - 12 \leq 0$

$(m+2)(m-6) \leq 0$

Therefore, $-2 \leq m \leq 6$

Substitute $a = 1, b = -m,\ c = m^2 - 3m - 9$ into the discriminant.

Reverse the inequality sign.



3 Let D be the discriminant of $x^2 + kx + 3 - k = 0$

$D = k^2 - 4(3-k)$

$= k^2 + 4k - 12$

$=(k-2)(k+6)$

[1] When $D > 0$, i.e., $(k-2)(k+6) > 0$:

Solving this gives: when $k <- 6$ *or* $k > 2$, it has two distinct real solutions.

[2] When $D = 0$, i.e., $(k-2)(k+6) = 0$:

Solving this gives: when $k =- 6, 2$, it has a repeated solution.

[3] When $D < 0$, i.e., $(k-2)(k+6) < 0$:

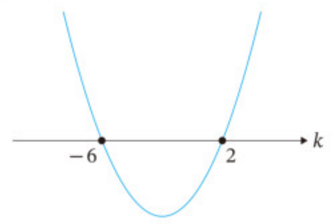
Solving this gives: when $-6 < k < 2$, it has two distinct imaginary solutions.

Summarising the above:

When $k < -6$ or $k > 2$, it has two distinct real solutions.

When $k = -6,\ 2$, it has a repeated solution.

When $-6 < k < 2$, it has two distinct imaginary solutions.



Try

Solve the following problems.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

1 Determine the nature of the solutions for the following quadratic equations.

(a) $3x^2 - 5x + 4 = 0$ **(b)** $4x^2 - 7x - 2 = 0$ **(c)** $25x^2 - 10x + 1 = 0$

2 Find the range of values for the constant m for which the quadratic equation $x^2 - 2mx + m + 2 = 0$ has two distinct imaginary solutions.

3 Let a be a real constant. Determine the nature of the solutions for the quadratic equation $x^2 - ax + 2a - 3 = 0$

Exercise

Solve the following problems.

1 Determine the nature of the solutions for the following quadratic equations.

(a) $x^2 - 5x + 3 = 0$ **(b)** $9x^2 - 12x + 4 = 0$ **(c)** $4x^2 - 8x + 5 = 0$

(d) $4x^2 - 4x + 1 = 0$ **(e)** $x^2 - 3x + 5 = 0$ **(f)** $x^2 - x - 2 = 0$

2 Solve the following problems.

(a) Find the value(s) of the constant a for which the quadratic equation $x^2 - (a+1)x + 4 = 0$ has a repeated solution.

(b) Find the range of values for the constant m for which the quadratic equation $x^2 + (m+6)x - 2m = 0$ has two distinct imaginary solutions.

(c) Find the range of values for the constant m for which the quadratic equation $x^2 - 2mx + 2m^2 - 3m = 0$ has real solutions.

★ Challenge

Solve the following problems.

(a) Let a be a real constant. Determine the nature of the solutions for the quadratic equation $x^2 + ax - a = 0$

(b) Let k be a real constant. Determine the nature of the solutions for the quadratic equation $x^2 - (k+2)x + k^2 = 0$

(c) Let a be a real constant. Determine the nature of the solutions for the quadratic equation $x^2 - 2(a-1)x + a^2 - 3a + 4 = 0$

3-7
MATHEMATICS

Relationship between Roots and Coefficients (1)

Chapter 3 Complex Numbers and Equations

Point!

! Relationship between Roots and Coefficients

When the two solutions (roots) of $ax^2+bx+c=0$ are α and β,

$$\alpha+\beta=-\frac{b}{a}$$

$$\alpha\beta=\frac{c}{a}$$

! For values of expressions involving solutions of quadratic equations, first find the values of $\alpha+\beta$ and $\alpha\beta$.

Rewrite the given expression in terms of $\alpha+\beta$ and $\alpha\beta$, and solve using the relationship between roots and coefficients.

In particular, the following transformations are often used.

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

Learning Outcomes

Deduce mathematical relationships between the roots of a quadratic equation and its coefficients, and use them to solve math problems.

Misconceptions

Confusing sum and product
Some students swap them, thinking the product is $-b/a$ and the sum is c/a.

Ignoring the coefficient a
Students sometimes assume formulas are just $-b$ and c, forgetting to divide by a.

Warm Up

Solve the following problems.

1 Find the sum and product of the two solutions of the quadratic equation $3x^2-6x-5=0$

2 Let α and β be the two solutions of the quadratic equation $2x^2-6x-3=0$

Find the value of the following expressions.

(a) $\alpha^2+\beta^2$ **(b)** $\alpha^3+\beta^3$ **(c)** $\frac{2}{\alpha}+\frac{2}{\beta}$ **(d)** $(\alpha+1)(\beta+1)$

Explanation

1 $3x^2-6x-5=0$

From the relationship between roots and coefficients, sum of two solutions is $-\frac{(-6)}{3}=2$

$\alpha+\beta=-\frac{b}{a}$

Product of two solutions is $\frac{(-5)}{3}=-\frac{5}{3}$

$\alpha\beta=\frac{c}{a}$

2 $2x^2 - 6x - 3 = 0$

$\alpha + \beta = 3,\ \alpha\beta = -\dfrac{3}{2}$

First find the values of $\alpha + \beta$ and $\alpha\beta$

(a) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 3^2 - 2 \times \left(-\dfrac{3}{2}\right)$

$= 9 + 3$

$= 12$

(b) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= 3^3 - 3 \times \left(-\dfrac{3}{2}\right) \times 3$

$= 27 + \dfrac{27}{2}$

$= \dfrac{81}{2}$

(c) $\dfrac{2}{\alpha} + \dfrac{2}{\beta} = \dfrac{2\beta + 2\alpha}{\alpha\beta}$

$= \dfrac{2(\alpha + \beta)}{\alpha\beta}$

$= \dfrac{2 \times 3}{-\frac{3}{2}}$

Multiply both the numerator and the denominator by 2

$= -4$

(d) $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1$

$= -\dfrac{3}{2} + 3 + 1$

$= \dfrac{5}{2}$

Try

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Solve the following problems.

1 Find the sum and product of the two solutions of the quadratic equation $4x^2 + 2x - 7 = 0$

2 Let α and β be the two solutions of the quadratic equation $2x^2 - 4x + 3 = 0$ Find the value of the following expressions.

(a) $\alpha^2 + \beta^2$ **(b)** $\alpha^3 + \beta^3$ **(c)** $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$ **(d)** $(\alpha - 1)(\beta - 1)$

Exercise

Solve the following problems.

1 Find the sum and product of the two solutions of the following quadratic equations.

(a) $x^2 + x + 2 = 0$ **(b)** $3x^2 - 7x - 6 = 0$ **(c)** $5x^2 - 3 = 0$

2 Let α and β be the two solutions of the quadratic equation $x^2 - 2x + 3 = 0$

Find the value of the following expressions.

(a) $\alpha^2 + \beta^2$ **(b)** $\alpha^3 + \beta^3$ **(c)** $\frac{1}{\alpha} + \frac{1}{\beta}$ **(d)** $(\alpha + 2)(\beta + 2)$

3 Let α and β be the two solutions of the quadratic equation $3x^2 + 6x - 1 = 0$

Find the value of the following expressions.

(a) $\alpha^2 + \beta^2$ **(b)** $\alpha^3 + \beta^3$ **(c)** $\frac{1}{\alpha} + \frac{1}{\beta}$ **(d)** $(\alpha - 2)(\beta - 2)$

★ Challenge

Let α and β be the two solutions of the quadratic equation $2x^2 - 6x + 3 = 0$ Find the value of the following expressions.

(a) $\alpha^2\beta + \alpha\beta^2$ **(b)** $(\alpha - \beta)^2$

(c) $\alpha^3 + \beta^3$ **(d)** $\frac{\beta^2}{\alpha} + \frac{\alpha^2}{\beta}$

3-8

MATHEMATICS

Relationship between Roots and Coefficients (2)

(Relationship between Two Roots)

Chapter 3 Complex Numbers and Equations

Point!

! For problems where the relationship between two solutions is known,

let α be one solution, and for the other,

express it in terms of α

Learning Outcomes

Deduce mathematical relationships between the roots of a quadratic equation and its coefficients, and use them to solve math problems.

Warm Up

If the difference between the two solutions of the quadratic equation $x^2 - (m-1)x - 6 = 0$ is 5, find the value of the constant m and the two solutions.

Explanation

Let one solution of $x^2 - (m-1)x - 6 = 0$ be α. The other solution can be expressed as $\alpha + 5$

From the relationship between roots and coefficients:

$$\begin{cases} \alpha + (\alpha + 5) = m - 1 \text{......(i)} \\ \alpha(\alpha + 5) = -6 \text{......(ii)} \end{cases}$$

Solving (ii) gives $\alpha = -2,\ -3$

When $\alpha = -2$, from (i), $m = 2$

Substitute $\alpha = -2$ into (i) to solve.

Also, the other solution in this case is $\alpha + 5 = -2 + 5 = 3$

When $\alpha = -3$, from (i), $m = 0$

Substitute $\alpha = -3$ into (i) to solve.

Also, the other solution in this case is $\alpha + 5 = -3 + 5 = 2$

Therefore,

When $m = 2$, the two solutions are $-2,\ 3$

When $m = 0$, the two solutions are $-3,\ 2$

Try

Solve the following problems.

1 If one solution of the quadratic equation $x^2 - 8x + k = 0$ is 3 times the other solution, find the value of the constant k and the two solutions.

2 If the difference between the two solutions of the quadratic equation $x^2 - (m + 1)x + 2 = 0$ is 1, find the value of the constant m and the two solutions.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 If one solution of the quadratic equation $x^2 + 6x + m = 0$ is 2 times the other solution, find the value of the constant m and the two solutions.

2 If one solution of the quadratic equation $8x^2 - mx + 1 = 0$ is 2 times the other solution, find the value of the constant m and the two solutions.

3 If the difference between the two solutions of the quadratic equation $x^2 + 6x + m = 0$ is 4, find the value of the constant m and the two solutions.

4 If the difference between the two solutions of the quadratic equation $4x^2 + mx + 3 = 0$ is 1, find the value of the constant m and the two solutions.

5 If the ratio of the two solutions of the quadratic equation $x^2 - 2x + m = 0$ is $2 : 3$, find the value of the constant m and the two solutions.

6 If one solution of the quadratic equation $x^2 - 6x + k = 0$ is the square of the other solution, find the value of the constant k and the two solutions.

★ Challenge

$x^2 + ax + bc = 0$ *and* $x^2 + bx + ca = 0$ have a non zero common root and $a \neq b$.
Show that the other two roots are roots of the quadratic equation $x^2 + cx + ab = 0$
$c \neq 0$

3-9
MATHEMATICS

Factorisation of Quadratic Expressions

Chapter 3 Complex Numbers and Equations

Point!

! Let α and β be the two solutions of $ax^2 + bx + c = 0$

It can be factorised as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Warm Up

Factorise the following expressions over the set of complex numbers.

(a) $2x^2 - 3x + 2$ **(b)** $x^4 + 2x^2 - 3$

Explanation

(a) Solving $2x^2 - 3x + 2 = 0$ using the quadratic formula gives $x = \dfrac{3 \pm i\sqrt{7}}{4}$

Therefore,

$$2x^2 - 3x + 2 = 2\left(x - \frac{3 + i\sqrt{7}}{4}\right)\left(x - \frac{3 - i\sqrt{7}}{4}\right)$$

(b) $x^4 + 2x^2 - 3 = \left(x^2 - 1\right)\left(x^2 + 3\right)$

$= (x + 1)(x - 1)\left(x^2 + 3\right)$

Here, solving $x^2 + 3 = 0$ gives $x = \pm i\sqrt{3}$

Therefore, $x^2 + 3 = \left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right)$

Thus, $x^4 + 2x^2 - 3 = \underline{(x + 1)(x - 1)\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right)}$

Try

Factorise the following expressions over the set of complex numbers.

(a) $x^2 + 2x + 5$ **(b)** $2x^2 - 6x + 1$ **(c)** $x^4 - x^2 - 6$

Exercise

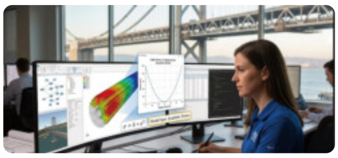
Factorise the following expressions over the set of complex numbers.

(a) $x^2 + 4x + 5$ **(b)** $x^2 + 4x + 2$ **(c)** $x^2 - 4x + 7$ **(d)** $2x^2 - x + 1$

(e) $3x^2 - 5x + 3$ **(f)** $2x^2 - 2x - 1$ **(g)** $x^4 - 9$ **(h)** $x^4 - 3x^2 + 2$

★ Challenge

An engineer is designing a suspension bridge and models the stress on one of the cables. The stress depends on the square of the displacement x^2,and the overall balance of forces is represented by the equation:



$$S(x) = 6x^4 - 7x^2 - 3$$

Analyze the equation to determine the possible values of x and explain what these values mean in the context of cable stability under load.

Learning Outcomes

Factorise quadratic expressions into their components within the realm of complex numbers.

Be careful not to forget the coefficient a of x^2

Misconceptions

Forgetting the coefficient of x^2

Students often try to split the middle term without considering the leading coefficient a.

Assuming all quadratics factorise neatly

Some believe every quadratic can be factorised into integers, ignoring cases where the discriminant is not a perfect square.

Be careful not to forget the coefficient 2 of x^2

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

3-10
MATHEMATICS

Quadratic Equations with Two Given Solutions

Chapter 3 Complex Numbers and Equations

Learning Outcomes

Form a quadratic equation when its roots are known.

Point!

One quadratic equation with two numbers $\alpha,\ \beta$ as solutions is

$x^2-(\alpha+\beta)x+\alpha\beta=0$

Warm Up

Solve the following problems.

1 Create one quadratic equation with integer coefficients that has the two numbers $\dfrac{1+i}{2}$ and $\dfrac{1-i}{2}$ as solutions.

2 Find two numbers whose sum is -1 and product is 2

3 Let $\alpha,\ \beta$ be the two solutions of the quadratic equation $x^2+2x-4=0$

Find a quadratic equation where the coefficient of x^2 is 1, having $\alpha+2$ and $\beta+2$ as its two solutions.

Explanation

1 For a quadratic equation with solutions $\dfrac{1+i}{2}$ and $\dfrac{1-i}{2}$

Sum of solutions: $\dfrac{1+i}{2}+\dfrac{1-i}{2}=1$

Product of solutions: $\dfrac{1+i}{2}\times\dfrac{1-i}{2}=\dfrac{1}{2}$

Therefore, one such equation is:

$x^2-x+\dfrac{1}{2}=0$

$\underline{2x^2-2x+1=0}$

One quadratic equation with α and β as its roots is $x^2-(\alpha+\beta)x+\alpha\beta=0$

Multiply both sides by 2 to make coefficients integers.

2 Create a quadratic equation whose solutions are the two numbers. Let the two numbers be $\alpha,\ \beta$.

$\alpha+\beta=-1$

$\alpha\beta=2$

One quadratic equation with α and β as its roots is $x^2-(\alpha+\beta)x+\alpha\beta=0$

In other words, the two numbers are the solutions of $x^2+x+2=0$

Solving this gives

$x=\dfrac{-1\pm\sqrt{7}i}{2}$ Therefore, the two numbers are $\dfrac{-1+\sqrt{7}i}{2},\ \dfrac{-1-\sqrt{7}i}{2}$

3 Since $\alpha,\ \beta$ are solutions of $x^2 + 2x - 4 = 0$, from the relationship between roots and coefficients:

$\alpha + \beta = -2 \ldots\ldots \text{(i)}$

$\alpha\beta = -4 \ldots\ldots \text{(ii)}$

Let the required quadratic equation be $x^2 + bx + c = 0$

Since the two solutions of this quadratic equation are $\alpha + 2,\ \beta + 2$, from the relationship between roots and coefficients:

$b = -\{(\alpha + 2) + (\beta + 2)\}$

$= -(\alpha + \beta + 4)$

$= -(-2 + 4)$

$= -2$

$c = (\alpha + 2)(\beta + 2)$

$= \alpha\beta + 2(\alpha + \beta) + 4$

$= -4 + 2 \times (-2) + 4$

$= -4$

Therefore, the required quadratic equation is $\underline{x^2 - 2x - 4 = 0}$

From (i), $\alpha + \beta = -2$

From (i) and (ii), $\alpha + \beta = -2,\ \alpha\beta = -4$

Try

Solve the following problems.

1 Create one quadratic equation that has the two numbers $3 + 2i$ and $3 - 2i$ as solutions.

2 Find two numbers where both the sum and the product are 1

3 Let $\alpha,\ \beta$ be the two solutions of the quadratic equation $x^2 + x - 3 = 0$

Find one quadratic equation that has α^2 and β^2 as its two solutions.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Create one quadratic equation that has the following two numbers as solutions.

(a) $-1,\ 2$ **(b)** $1+4i,\ 1-4i$ **(c)** $\dfrac{5+\sqrt{7}}{3},\ \dfrac{5-\sqrt{7}}{3}$

2 Find two numbers whose sum and product are as follows.

(a) Sum is 2, Product is 5 **(b)** Sum is 4, Product is 2

3 Let $\alpha,\ \beta$ be the two solutions of the quadratic equation $x^2+2x+3=0$

Find one quadratic equation that has $\alpha+3\beta$ and $3\alpha+\beta$ as its two solutions.

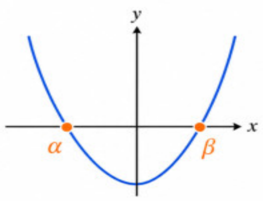
4 Let $\alpha,\ \beta$ be the two solutions of the quadratic equation $x^2-2x+7=0$

Find one quadratic equation that has α^2 and β^2 as its two solutions.

5 Let $\alpha,\ \beta$ be the two solutions of the quadratic equation $2x^2+3x+4=0$

Find one quadratic equation with integer coefficients that has $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ as its two solutions.

★ Challenge



In the opposite figure:
If α *and* β are the two roots of the quadratic equation. Create a quadratic equation whose two roots are:

$$\frac{1}{\alpha}\ and\ \frac{1}{\beta}$$

3-11
MATHEMATICS

Signs of Two Solutions

Chapter 3 Complex Numbers and Equations

Point!

! For problems where the signs of the two solutions of a quadratic equation are known, form and solve simultaneous inequalities based on the conditions.

(i) Two distinct positive solutions: $\Longrightarrow$ $D > 0$, $\alpha + \beta > 0$, $\alpha\beta > 0$

(ii) Two distinct negative solutions: $\Longrightarrow$ $D > 0$, $\alpha + \beta < 0$, $\alpha\beta > 0$

(iii) Solutions with different signs: $\Longrightarrow$ $\alpha\beta < 0$

Discriminant D is not needed.

Learning Outcomes

Deduce mathematical relationships between the roots of a quadratic equation and its coefficients, and use them to solve math problems or study the signs of the roots (solutions of the quadratic equation).

Warm Up

If the quadratic equation $x^2 - 2mx + m + 6 = 0$ has two distinct negative solutions, find the range of values for the real number m.

Explanation

Let α, β be the two solutions of this quadratic equation, and D be the discriminant.

From $D > 0$, $(-2m)^2 - 4 \times 1 \times (m+6) > 0$

$$4m^2 - 4m - 24 > 0$$

$$m^2 - m - 6 > 0$$

$$(m-3)(m+2) > 0$$

Therefore, $m < -2$ or $m > 3$......(i)

From $\alpha + \beta < 0$, $2m < 0$

Therefore, $m < 0$...... (ii)

From $\alpha\beta > 0$, $m + 6 > 0$

Therefore, $m > -6$...... (iii)

Finding the common range of (i), (ii), and (iii),

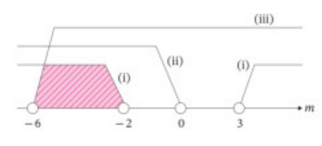
we get $-6 < m < -2$

Common Mistakes

From $\alpha + \beta < 0$, $-2m < 0$

Therefore, $m > 0$

You forgot the negative sign in the formula.



Try

Solve the following problems.

1 If the quadratic equation $x^2 + 2mx + m + 2 = 0$ has two distinct positive solutions, find the range of values for the real number m.

2 If the two solutions of the quadratic equation $x^2 - (m - 3)x - m - 4 = 0$ have different signs, find the range of values for the real number m.

Exercise

Solve the following problems.

1 If the quadratic equation $x^2 + 2(m - 1)x + 2m^2 - 5m - 3 = 0$ has two distinct positive solutions, find the range of values for the real number m.

2 If the two solutions of the quadratic equation $x^2 + 4kx - k - 3 = 0$ have different signs, find the range of values for the real number k.

3 If the quadratic equation $x^2 + kx + 2k + 5 = 0$ has two distinct negative solutions, find the range of values for the real number k.

★ Challenge

In a battery production factory, the efficiency factor x depends on a design parameter m. The relationship between efficiency and the design parameter is modeled by the quadratic equation:

$$x^2 + (m + 1)x + m + 4 = 0$$

If the equation has two distinct real solutions, analyze the range of values for m in each case:

(a) When both solutions are positive, representing successful efficiency levels of the battery.

(b) When both solutions are negative, representing failure in performance.

(c) When one solution is positive and the other is negative, representing opposite behaviors of the battery (success in one case and failure in another).

Reflect

How can analyzing the properties of an equation (such as knowing the number of its roots or their relation to the sum and product of the roots) without fully solving it help you understand mathematics better?

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

3-12

MATHEMATICS

Remainder Theorem (1)

Chapter 3 Complex Numbers and Equations

In economics and social sciences, researchers often use polynomial regression models to analyze and predict trends. For example, economists may study how unemployment rates change over time, or how income distribution shifts across different social groups. Polynomials provide a flexible way to approximate complex data patterns and forecast future outcomes. Remainder Theorem: Allows quick evaluation of the model at a specific year, such as predicting unemployment in year 5 by calculating P(5). Factor Theorem: Helps identify when the model reaches zero, for example, when poverty rates drop to zero. Synthetic Division: Provides a fast method to test values and simplify calculations when analyzing large datasets.

Learning Outcomes

Apply the Remainder Theorem to divide polynomials efficiently.

Use the Remainder Theorem to solve mathematical problems involving polynomial functions.

Guiding Question: How can polynomial theorems help economists and social scientists test values and predict future trends efficiently?

Point!

! Remainder Theorem

The remainder when polynomial $P\ (x)$ is divided by the linear expression $x - a$ is $\boldsymbol{P}\ (\boldsymbol{a})$

If $P\ (x) \div (x - a) = Q\ (x) + R$,

$P(x) = Q\ (x)\ (x - a) + R$.

Therefore, $P(a) = Q(a)\underbrace{(a - a)}_{\text{becomes } 0} + R$

$= R$

The remainder when a polynomial $P\ (x)$ is divided by the linear expression $ax - b$ is $\boldsymbol{P}\left(\dfrac{\boldsymbol{b}}{\boldsymbol{a}}\right)$

Example: Remainder when $P(x)$ is divided by $x - 4$ is $\boldsymbol{P}(4)$

Remainder when $P\ (x)$ is divided by $4x + 3$ is $\boldsymbol{P}\left(-\dfrac{3}{4}\right)$

Warm Up

Solve the following problems.

1 Find the remainder when $3x^4 - 8x^3 - 5x^2 + 12$ is divided by $3x - 2$

2 If the remainder when $x^3 + ax^2 + 3x + 1$ is divided by $x + 3$ is 4, find the value of a.

Explanation

1 Let $P\ (x) = 3x^4 - 8x^3 - 5x^2 + 12$ The remainder when divided by $3x - 2$ is:

Substitute $x = \dfrac{2}{3}$

$$\mathrm{P}\left(\frac{2}{3}\right) = 3 \times \left(\frac{2}{3}\right)^4 - 8 \times \left(\frac{2}{3}\right)^3 - 5 \times \left(\frac{2}{3}\right)^2 + 12$$

$$= \underline{8}$$

2 Let $P(x) = x^3 + ax^2 + 3x + 1$ The remainder when divided by $x + 3$ is:

$P(-3) = (-3)^3 + a \times (-3)^2 + 3 \times (-3) + 1$

$= 9a - 35$

Therefore, $9a - 35 = 4$

Solving this gives $a = \frac{13}{3}$

Substitute $x = -3$

Try

Solve the following problems.

1 Find the remainder when $x^3 - 5x^2 + 4x - 1$ is divided by $2x - 1$

2 If the remainder when $x^3 + ax^2 + 3x + 1$ is divided by $x - 3$ is 1, find the value of a.

Exercise

Solve the following problems.

1 Find the remainder when $x^3 + 2x^2 - 3x - 4$ is divided by $x + 2$

2 Find the remainder when $3x^3 + 2x^2 - 5x + 6$ is divided by $x - 1$

3 Find the remainder when $6x^3 - x^2 + x - 2$ is divided by $3x - 2$

4 Find the remainder when $4x^3 - 2x^2 - 4x + 3$ is divided by $2x + 1$

5 If the remainder when $x^3 - x^2 + ax - 4$ is divided by $x + 1$ is -2, find the value of a.

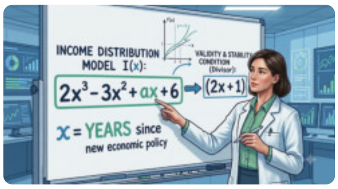
6 If the remainder when $x^3 + 4x^2 - 5x + a$ is divided by $x + 3$ is 8, find the value of a.

7 If $x^3 - 3x^2 + 4x + a$ is divisible by $x - 2$, find the value of a.

★ Challenge

A researcher models the **income distribution** of a community with the polynomial:

$$P(x) = 2x^3 - 3x^2 + ax + 6$$



where x represents the number of years since a new economic policy was introduced.

For the model to be **valid and stable**, it must be divisible by the condition:

$$(2x + 1)$$

which represents a fairness constraint in the system.

1. Find the value of a that ensures divisibility.
2. Discuss mathematically:

• What changes if the divisor were instead $(x + 2)$?

• How does the choice of divisor affect the possible values of a?

• Can a single polynomial be divisible by more than one linear factor at the same time? Under what conditions?

Misconceptions

Divisor confusion: Students may substitute the wrong value (e.g., use $x = 2$ instead of $x = \frac{2}{3}$ for $3x - 2$).

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

3-13

MATHEMATICS

Remainder Theorem (2)

Chapter 3 Complex Numbers and Equations

Point!

! If $P(x)$ is divided by a quadratic expression $(x-\alpha)(x-\beta)$,

Let the quotient be $Q(x)$ and the remainder be $ax+b$

Express it as $P(x) = (x-\alpha)(x-\beta)Q(x) + ax + b$

! $(x-\alpha)(x-\beta)Q(x)$ becomes 0 when $x=\alpha$ or $x=\beta$ is substituted.

Learning Outcomes

Apply the Remainder Theorem to divide polynomials efficiently.
Use the Remainder Theorem to solve mathematical problems involving polynomial functions.

The degree of the remainder must be lower than the degree of the divisor.

$A = B \times Q + R$

Warm Up

If a polynomial $P(x)$ is divided by $x-1$, the remainder is 3, and when divided by $x+3$, the remainder is -5. Find the remainder when $P(x)$ is divided by x^2+2x-3

Explanation

Let the quotient be $Q(x)$ and the remainder be $ax+b$ when $P(x)$ is divided by x^2+2x-3

$P(x) = (x^2+2x-3)Q(x) + ax + b$

Factorise the quadratic divisor.

$P(x) = (x-1)(x+3)Q(x) + ax + b$

Since the remainder when divided by $x-1$ is 3, $P(1) = 3$

$P(1) = \underbrace{(1-1)(1+3)Q(1)}_{=0} + a + b$

$= a + b$

$a + b = 3$......(i)

Since the remainder when divided by $x+3$ is -5, $P(-3) = -5$

$P(-3) = \underbrace{(-3-1)(-3+3)Q(-3)}_{=0} - 3a + b$

$= -3a + b$

$-3a + b = -5$......(ii)

Solving the simultaneous equations (i) and (ii) gives $a = 2, b = 1$

Therefore, the required remainder is $2x+1$

Try

If a polynomial $P(x)$ is divided by $x - 2$, the remainder is 3, and when divided by $x + 3$, the remainder is −7 Find the remainder when $P\ (x)$ is divided by $x^2 + x - 6$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 If a polynomial $P\ (x)$ is divided by $x - 1$, the remainder is 5, and when divided by $x + 2$, the remainder is −1 Find the remainder when $P(x)$ is divided by $(x - 1)(x + 2)$.

2 If a polynomial $P\ (x)$ is divided by $x - 2$, the remainder is 4, and when divided by $x + 3$, the remainder is −11 Find the remainder when $P(x)$ is divided by $x^2 + x - 6$

★ Challenge

A polynomial $P\ (x)$ is divisible by $x - 1$, and leaves a remainder of -4 when divided by $x + 3$ Find the remainder when $P\ (x)$ is divided by $x^2 + 2x - 3$

3-14

MATHEMATICS

Remainder Theorem (3)

Chapter 3 Complex Numbers and Equations

Point!

! If $P(x)$ is divided by a quadratic expression $(x-\alpha)(x-\beta)$,

Let the quotient be $Q(x)$ and the remainder be $ax+b$

Express it as $P(x) = (x-\alpha)(x-\beta)Q(x) + ax + b$

Learning Outcomes

Apply the Remainder Theorem to divide polynomials efficiently.
Use the Remainder Theorem to solve mathematical problems involving polynomial functions.

The degree of the remainder must be lower than the degree of the divisor.

Refer to **1-5** $A = B \times Q + R$

Warm Up

A polynomial $P(x)$ leaves a remainder of $7x+6$ when divided by x^2-x-6, and $4x-3$ when divided by x^2+2x-3

Find the remainder when $P(x)$ is divided by x^2+x-2

Explanation

Let the quotient be $Q(x)$ and the remainder be $ax+b$ when $P(x)$ is divided by x^2+x-2

$P(x) = \left(x^2+x-2\right)Q(x) + ax + b$

$\quad = (x+2)(x-1)Q(x) + ax + b$......(i)

Let the quotient be $Q_1(x)$ when $P(x)$ is divided by x^2-x-6

Since the remainder is $7x+6$,

$P(x) = \left(x^2-x-6\right)Q_1(x) + 7x + 6$

$\quad = (x+2)(x-3)Q_1(x) + 7x + 6$

Here, $P(-2) = -14 + 6 = -8$ Also from (i), $P(-2) = -2a + b$

Thus, $-2a + b = -8$...... (ii)

Similarly, let the quotient be $Q_2(x)$ when $P(x)$ is divided by x^2+2x-3 Since the remainder is $4x-3$,

$P(x) = \left(x^2+2x-3\right)Q_2(x) + 4x - 3$

$\quad = (x-1)(x+3)Q_2(x) + 4x - 3$

Here, $P(1) = 4 - 3 = 1$ Also from (i), $P(1) = a + b$. Thus, $a + b = 1$...... (iii)

Solving the simultaneous equations (ii) and (iii) gives $a = 3$, $b = -2$

Therefore, the required remainder is $3x-2$

Try

A polynomial $P(x)$ leaves a remainder of $3x + 2$ when divided by $x^2 - x - 2$, and $7x - 6$ when divided by $x^2 + 2x - 3$

Find the remainder when $P(x)$ is divided by $x^2 - 1$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1. A polynomial $P(x)$ leaves a remainder of $2x - 2$ when divided by $x^2 - 2x - 3$, and $-x + 7$ when divided by $x^2 - 4$

   Find the remainder when $P(x)$ is divided by $x^2 - x - 2$

2. A polynomial $P(x)$ leaves a remainder of $-x + 4$ when divided by $x^2 - 3x + 2$, and $3x$ when divided by $x^2 - 4x + 3$

   Find the remainder when $P(x)$ is divided by $x^2 - 5x + 6$

3. A polynomial $P(x)$ leaves a remainder of $x - 5$ when divided by $(x+1)^2$, and 6 when divided by $x - 2$

   In this case, find the remainder when $P(x)$ is divided by $(x + 1)^2(x - 2)$

★ Challenge

Evaluate exactly what the remainder will be when $x^{57} + 57$ is divided by $x + 1$

3-15

MATHEMATICS

Synthetic Division

Chapter 3 Complex Numbers and Equations

Point!

! When dividing a polynomial by a linear expression $x - a$, using Synthetic Division makes it easy to find the quotient and remainder.

Learning Outcomes

Use synthetic division to simplify and divide polynomials.

Misconceptions

Forgetting to insert a zero coefficient for missing terms (like the x^2 term)

Mixing up the quotient coefficients with the remainder.

Warm Up

Using Synthetic Division, find the quotient and remainder when $x^3 + 5x + 11$ is divided by $x + 2$

Explanation

Steps for Synthetic Division

1. Extract the coefficients of $x^3 + 5x + 11$

Since there is no x^2 term, the coefficient is 0

$$\begin{array}{cccc} 1 & 0 & 5 & 11 \end{array}$$

2. Since dividing by $x + 2$, write -2

When dividing by $x - a$, write the value of a on the left.

$$\begin{array}{r|cccc} -2 & 1 & 0 & 5 & 11 \end{array}$$

3. Bring down the first number 1 as is, multiply by -2 written on the left, and write the answer in the second row.

$$\begin{array}{r|cccc} -2 & 1 & 0 & 5 & 11 \\ & & -2 & & \\ \hline & 1 & & & \end{array}$$

4. Add the two numbers vertically.

$$\begin{array}{r|cccc} -2 & 1 & 0 & 5 & 11 \\ & & -2 & & \\ \hline & 1 & -2 & & \end{array}$$

5. Repeat steps 3 and 4 until the right end of the 3rd row is filled.

$$\begin{array}{r|cccc} -2 & 1 & 0 & 5 & 11 \\ & & -2 & 4 & -18 \\ \hline & 1 & -2 & 9 & \boxed{-7} \end{array}$$

From right to left: Remainder, Constant term of Quotient, Coefficient of x, Coefficient of x^2, ..., Coefficient of x^n.

Therefore, the required quotient and remainder are:

Quotient $x^2 - 2x + 9$, Remainder -7

Try

Using Synthetic Division, find the quotient and remainder when A is divided by B for the following polynomials $A$and B.

(a) $A = x^3 + 5x - 6,\ B = x - 1$ **(b)** $A = 3x^3 - 5x^2 - 3x - 1,\ B = x - 3$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Using Synthetic Division, find the quotient and remainder when A is divided by B for the following polynomials A and B.

(a) $A = x^3 - 5x + 6,\ B = x - 2$ **(b)** $A = x^3 + 2x^2 - x - 3,\ B = x - 1$

(c) $A = 4x^2 - 2x + 2,\ B = x - 1$ **(d)** $A = 3x^4 + 5x^3 - 2x - 12,\ B = x + 2$

★ Challenge

The polynomial $P(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$ has a horizontal intercept at $x = \frac{1}{2}$ with multiplicity 2. Find the other intercepts of $P(x)$.

3-16
MATHEMATICS

Factor Theorem

Chapter 3 Complex Numbers and Equations

Point!

! Factorisation using the Factor Theorem

If a polynomial $P(x)$ is divisible by the linear expression $x-a$, it can be factorised as $P(x) = (x-a)Q(x)$.

! In factorisation using the Factor Theorem, find a value a such that $P(a) = 0$

Values for a are easy to find by substituting **±(divisors of the constant term)** in order.

Example: Divisors of the constant term 6 in x^3+3x^2+2x+6 are $1,\ 2,\ 3,\ 6$

$\Rightarrow$ Substitute $\pm1,\ \pm2,\ \pm3,\ \pm6$ in order.

Learning Outcomes

Apply the factor theorem to factorise polynomials.

$Q(x)$ is the quotient when $P(x)$ is divided by $x-a$.

Warm Up

Factorise the following expressions.

(a) x^3+2x^2-x-2 **(b)** $x^4-6x^3+8x^2+8x-16$

Explanation

(a) Let $P(x) = x^3+2x^2-x-2$

$P(1) = 1^3+2\times1^2-1-2$

$= 0$

Dividing $P(x)$ by $x-1$,

$P(x) = (x-1)\underline{(x^2+3x+2)}$

$= (x-1)(x+1)(x+2)$

Since the constant term is -2, substitute $\pm1,\ \pm2$ in order.

Can be factorised as $P(x) = (x-1)(\dots)$.

Use synthetic division

1	1	2	−1	−2
		1	3	2
	1	3	2	0

Factorise the quadratic part.

(b) Let $P(x) = x^4 - 6x^3 + 8x^2 + 8x - 16$

$P(2) = 2^4 - 6 \times 2^3 + 8 \times 2^2 + 8 \times 2 - 16$

$= 0$

Dividing $P(x)$ by $x - 2$,

$P(x) = (x - 2)(x^3 - 4x^2 + 8)$

Let $Q(x) = x^3 - 4x^2 + 8$

$Q(2) = 2^3 - 4 \times 2^2 + 8$

$= 0$

Dividing $Q(x)$ by $x - 2$,

$Q(x) = (x - 2)(x^2 - 2x - 4)$

Therefore, $P(x) = (x - 2)\underbrace{(x - 2)(x^2 - 2x - 4)}_{Q(x)}$

$= (x - 2)^2 (x^2 - 2x - 4)$

Substitute $\pm 1, \pm 2, \pm 4$... in order.

2	1	−6	8	8	−16
		2	−8	0	16
	1	−4	0	8	0

Factorise the cubic part using the Factor Theorem.

Substitute $\pm 1, \pm 2, \pm 4$... in order.

2	1	−4	0	8
		2	−4	−8
	1	−2	−4	0

The quadratic part cannot be factorised further, so answer as is.

Try

Factorise the following expressions.

(a) $x^3 - 2x^2 - 5x + 6$ **(b)** $x^3 - x^2 - 16x - 20$

(c) $2x^3 - x^2 - 5x - 2$ **(d)** $2x^3 - 3x^2 - x - 2$

(e) $x^4 + 3x^3 - 27x^2 + 13x + 42$ **(f)** $x^4 - 4x^3 + 8x^2 - 8x + 3$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Factorise the following expressions.

(a) $x^3 - 7x - 6$ **(b)** $x^3 + 4x^2 + x - 6$

(c) $x^3 + 6x^2 + 11x + 6$ **(d)** $x^3 - 3x + 2$

(e) $x^3 + x^2 - 8x - 12$ **(f)** $x^3 - 3x^2 - 9x - 5$

(g) $2x^3 - 3x^2 - 3x + 2$ **(h)** $3x^3 + x^2 - 8x + 4$

(i) $2x^3 + 7x^2 + 8x + 3$ **(j)** $2x^3 - 7x^2 + 3x + 6$

(k) $x^3 + 4x^2 + 2x - 4$ **(l)** $x^3 - 3x^2 + 2$

(m) $x^4 - 10x^3 + 35x^2 - 50x + 24$ **(n)** $x^4 - 5x^3 - 3x^2 + 17x - 10$

(o) $2x^4 - 5x^3 - 20x^2 + 20x + 48$ **(p)** $x^4 - 4x^3 - 4x^2 + 13x - 6$

(q) $x^4 - 4x^3 + 3x^2 + 2x - 6$ **(r)** $x^4 - x^3 + x^2 + x - 2$

★ Challenge

Let $P(x)$ be a polynomial defined as:

$$P(x) = x^4 - 3x^3 + ax^2 + bx + 12$$

Given that both $(x - 1)$ *and* $(x + 2)$ are factors of $P(x)$:

1. Find the exact values of the constants a *and* b.
2. Hence, determine all the roots of the equation $P(x) = 0$

Reflect

How does breaking down polynomials into factors help us understand both mathematics and real-world systems more deeply?

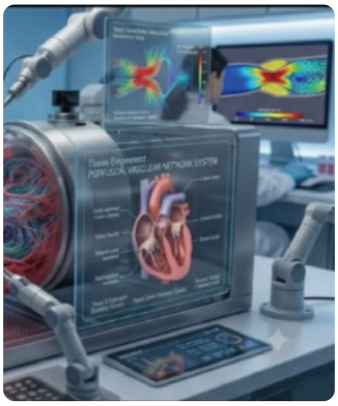
3-17

MATHEMATICS

High-Degree Equations

Chapter 3 Complex Numbers and Equations

Concept 3 · Higher- degree equations



In biomedical engineering, modeling physical structures requires moving beyond simple flat lines or parabolas. When researchers design synthetic heart valves or model the blood flow velocity through a constricted artery, the fluid dynamics change in complex, non-linear ways. These phenomena are modeled using cubic and higher-degree polynomial equations.

Unlike quadratic equations, which only model a single peak or valley, a cubic equation can capture a system that starts at a baseline, experiences a sudden surge (like a heartbeat), and stabilizes back to equilibrium.

Guiding Question: How do the algebraic properties of cubic equations—such as multiple turning points, repeated roots, and cube roots of unity—enable us to model systems more accurately than quadratic equations?

Learning Outcomes

Solve higher-degree equations.

Point!

! An n—th degree equation **always has** n solutions when considered within the range of complex numbers (excluding cases with repeated solutions).

! How to solve Cubic and Quartic Equations

❶ Factorise

- For cubic equations: If it can be transformed into $x^3 + a^3 = 0 \Rightarrow$

  Use $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

  : Otherwise $\Rightarrow$ Use the **Factor Theorem**.

- For quartic equations: If there are only $x^4,\ x^2$, and constant terms $\Rightarrow$

  Treat **x^2 as a chunk**.

  : Otherwise $\Rightarrow$ Use the **Factor Theorem**.

❷ Find the solutions

- The solutions for $(x-\alpha)(x-\beta)(x-\gamma) = 0$ are $x = \alpha$ or β or γ
- For $(x-a)\left(ax^2 + bx + c\right) = 0$, if $ax^2 + bx + c$ cannot be factorised $\Rightarrow$

  Use the **Quadratic Formula**.

  The solutions in this case are $x = a$ or $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Warm Up

Solve the following equations.

(a) $x^3-8=0$ **(b)** $x^3-2x^2-5x+6=0$

(c) $x^4-7x^2-18=0$ **(d)** $x^4-6x^3+8x^2+8x-16=0$

Explanation

(a) $x^3-8=0$

Use $x^3+a^3=(x+a)\left(x^2-ax+a^2\right)$

$(x-2)\left(x^2+2x+4\right)=0$

Since x^2+2x+4 cannot be factorised, find solutions using the quadratic formula.

Therefore, the required solutions are $\underline{x=2,\ -1\pm\sqrt{3}i}$

(b) $x^3-2x^2-5x+6=0$

Since formulas cannot be used, use the Factor Theorem.

$(x-1)\left(x^2-x-6\right)=0$

Factorise the quadratic part.

$(x-1)(x-3)(x+2)=0$

Therefore, the required solutions are $\underline{x=1,\ 3,\ -2}$

(c) $x^4-7x^2-18=0$

Since it's a quartic equation with only x^4, x^2, and constant terms, factorise treating x^2 as a chunk.

$\left(x^2\right)^2-7x^2-18=0$

$\left(x^2-9\right)\left(x^2+2\right)=0$

$x^2=9$, or $x^2=-2$

Solve each quadratic equation.

Therefore, the required solutions are $\underline{x=\pm3,\ \pm\sqrt{2}i}$

(d) $x^4-6x^3+8x^2+8x-16=0$

Use the Factor Theorem.

$(x-2)\left(x^3-4x^2+8\right)=0$

Factorise the cubic part. Since formulas cannot be used, use the Factor Theorem.

$(x-2)^2\left(x^2-2x-4\right)=0$

Since x^2-2x-4 cannot be factorised, find solutions using the quadratic formula.

Therefore, the required solutions are $\underline{x=2,\ 1\pm\sqrt{5}}$

Try

Solve the following equations.

(a) $x^3=-1$ **(b)** $3x^3+x^2-8x+4=0$

(c) $2x^3-3x^2-x-2=0$ **(d)** $x^3-x^2-16x-20=0$

(e) $x^4-5x^2+4=0$ **(f)** $x^4-10x^3+35x^2-50x+24=0$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following equations.

(a) $x^3 = 27$

(b) $x^3 = -125$

(c) $8x^3 = 1$

(d) $x^3 + 6x^2 + 11x + 6 = 0$

(e) $x^3 + 3x^2 - 4x - 12 = 0$

(f) $3x^3 - 2x^2 - 3x + 2 = 0$

(g) $x^3 - x^2 - 4 = 0$

(h) $x^3 + 4x^2 + 2x - 4 = 0$

(i) $x^3 - 4x^2 + 9x - 10 = 0$

(j) $x^3 + x^2 - 8x - 12 = 0$

(k) $2x^3 + 7x^2 + 8x + 3 = 0$

(l) $x^3 + 6x^2 + 12x + 8 = 0$

(m) $x^4 - 5x^2 - 36 = 0$

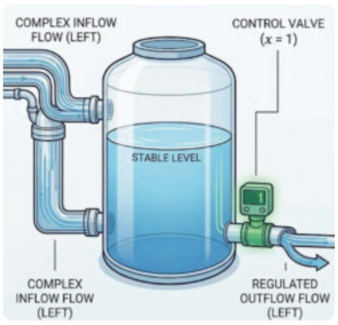
(n) $x^4 - 6x^2 - 27 = 0$

(o) $x^4 - x^2 - 2 = 0$

(p) $x^4 - 9x^3 + 29x^2 - 39x + 18 = 0$

(q) $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$

★ Challenge



You have a household water tank. Water flows into the tank at different rates and flows out at other rates, all depending on the pressure level x:

Inflow (increasing water):

$x^4 + x^2 + x$

Outflow (decreasing water):

$x^3 + 2$

For the water level in the tank to remain perfectly stable.

1-Write the equation that expresses the balance between inflow and outflow.

2-Solve the equation to find the values of x that keep the water level stable.

3-Determine whether the solutions are only real numbers or if complex solutions also exist.

3-18
MATHEMATICS

Determination of Cubic Equation

Chapter 3 Complex Numbers and Equations

Point!

! When a solution to a cubic equation is given, calculate by substituting the solution into the given equation .

Learning Outcomes

Solve higher-degree equations and formulate cubic equations based on given data.

Warm Up

If the cubic equation $x^3 + ax^2 + bx + 5 = 0$ has $2 + i$ as a solution, find the values of real numbers a, b. Also, find the other solutions.

Explanation

Since $x = 2 + i$ is a solution,

$(2+i)^3 + a(2+i)^2 + b(2+i) + 5 = 0$

$(3a + 2b + 7) + (4a + b + 11)i = 0$

Therefore, $\begin{cases} 3a + 2b + 7 = 0 \\ 4a + b + 11 = 0 \end{cases}$

Solving this gives $a = -3, b = 1$

In this case, the original equation is $x^3 - 3x^2 + x + 5 = 0$

Factorising the LHS gives $(x+1)\left(x^2 - 4x + 5\right) = 0$

Therefore, $x = -1$ or $2 \pm i$

Summarising the above:

$a = -3, b = 1$, and the other solutions are $x = -1$ or $2 - i$

Substitute the solution.

Convert to $a + bi$ form.

Compare real and imaginary parts of both sides (see **3-1**).

Since formulas cannot be used, use the Factor Theorem.

Do not include $2 + i$ in the answer.

Try

Solve the following problems.

1 If the cubic equation $x^3 + 4x^2 + ax + b = 0$ has −3 and 1 as solutions, find the values of real numbers a, b. Also, find the other solutions.

2 If the cubic equation $x^3 + ax^2 + bx - 4 = 0$ has $1 + i$ as a solution, find the values of real numbers $a,\ b$. Also, find the other solutions.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 If the cubic equation $x^3 - ax^2 - x + b = 0$ has −1 and 2 as solutions, find the values of real numbers a, b. Also, find the other solutions.

2 If the cubic equation $x^3 + ax^2 + bx - 8 = 0$ has 1 and 2 as solutions, find the values of real numbers a, b. Also, find the other solutions.

3 If one solution of the cubic equation $x^3 - kx^2 + (3k - 1)x - 24 = 0$ is 2, find the value of the real number k. Also, find the other solutions.

4 If the cubic equation $x^3 + ax^2 + bx - 20 = 0$ has $1 - 3i$ as a solution, find the values of real numbers a, b. Also, find the other solutions.

5 If the cubic equation $x^3 + ax^2 + bx + 10 = 0$ has $1 + 2i$ as a solution, find the values of real numbers a, b. Also, find the other solutions.

★ Challenge

If the cubic equation $x^3 + ax + b = 0$ has $1 + \sqrt{2}i$ as a solution, find the values of real numbers a, b. Also, find the other solutions.

3-19

MATHEMATICS

Cube Roots of Unity

Chapter 3 Complex Numbers and Equations

Learning Outcomes

Find the cube roots of unity and apply their properties in simplifying algebraic expressions.

Point!

! A number that becomes 1 when cubed is called **a cube root of unity**. One of these that is a complex not real number is denoted by ω(omega).

! ω has the following properties:

$\omega^3 = 1$

$\omega^2 + \omega + 1 = 0$

Since $\omega^3 = 1$, $\omega^3 - 1 = 0 \rightarrow$
$(\omega - 1)(\omega^2 + \omega + 1) = 0$
Since $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$

! In calculations using ω, first use $\omega^3 = 1$ to lower the degree of ω. After that, if ω^2 and ω remain, use $\omega^2 + \omega + 1 = 0$ to find the value of the expression.

Warm Up

Let ω be one of the imaginary cube roots of unity. Find the value of the following expressions.

(a) ω^9 **(b)** $\omega^4 + \omega^5$ **(c)** $\omega + \dfrac{1}{\omega}$

★ Challenge

Evaluate the value of the following expression without expanding the brackets directly:

$$(1 - \omega + \omega^2)(1 + \omega - \omega^2)$$

Explanation

(a) $\omega^9 = (\omega^3)^3 = 1^3 = 1$

Create ω^3 to use $\omega^3 = 1$

(b) $\omega^4 + \omega^5 = \omega \times \omega^3 + \omega^2 \times \omega^3 = \omega + \omega^2 = -1$

Create ω^3 to use $\omega^3 = 1$

Since $\omega^2 + \omega + 1 = 0$, $\omega + \omega^2 = -1$

(c) $\omega + \dfrac{1}{\omega} = \dfrac{\omega^2 + 1}{\omega} = \dfrac{-\omega}{\omega} = -1$

Since $\omega^2 + \omega + 1 = 0$, $\omega^2 + 1 = -\omega$

Try

Let ω be one of the imaginary cube roots of unity. Find the value of the following expressions.

(a) ω^{2010} **(b)** $\omega^2 + \omega$ **(c)** $(1 + \omega)(1 + \omega^2)$

Exercise

1 Let ω be one of the imaginary cube roots of unity. Find the value of the following expressions.

(a) ω^3 **(b)** $\omega^2 + \omega^4 + \omega^6$ **(c)** $\dfrac{1}{\omega} + \dfrac{1}{\omega^2} + 1$

3-20 Cubic Equation with Repeated Roots

MATHEMATICS

Chapter 3 Complex Numbers and Equations

Point!

! The problem of determining an equation so that a cubic equation has a repeated root is solved by first using the **factor theorem** to factor the given cubic expression into the form $(x-a)\left(x^2+bx+c\right)$.

! There are two cases where $(x-a)\left(x^2+bx+c\right)=0$ has a repeated root.

- If $x^2+bx+c=0$ has $x=a$ as a solution:

$\Longrightarrow$ **Substitute $x=a$ into $x^2+bx+c=0$**

- If $x^2+bx+c=0$ has a repeated root:

$\Longrightarrow$ **Find the condition where the discriminant D of $x^2+bx+c=0$ becomes 0**

Learning Outcomes

Solve cubic equations with repeated roots.

Warm Up

Find the value of the constant a such that $x^3-2x^2+(a-3)x+a=0$ has a repeated root.

Explanation

Let $P(x)=x^3-2x^2+(a-3)x+a$......(i)

Factorise $x^3-2x^2+(a-3)x+a$ using the Factor Theorem.

$P(-1)=0$

Therefore, $P(x)=(x+1)\left(x^2-3x+a\right)$

$(x+1)\left(x^2-3x+a\right)=0$ has a repeated root when:

$x^2-3x+a=0$ has $x=-1$ as a solution, or $x^2-3x+a=0$ has a repeated root.

[1] Substitute $x=-1$ into $x^2-3x+a=0$

If $x^2-3x+a=0$ has $x=-1$ as a solution:

$(-1)^2-3\times(-1)+a=0$

Solving this gives $a=-4$

In this case, (i) becomes $(x+1)^2(x-4)=0$, which satisfies the condition.

[2] When $x^2-3x+a=0$ has a repeated root, let D be the discriminant of this expression:

$D=(-3)^2-4\times1\times a=0$

Solving this gives $a=\dfrac{9}{4}$

In this case, (i) becomes $(x+1)\left(x-\dfrac{3}{2}\right)^2=0$, which satisfies the condition.

Therefore, the required values of a are $a=-4,\ \dfrac{9}{4}$

Try

Find the value of the constant a such that $x^3 - x^2 + (a-2)x + a = 0$ has a repeated root.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 Find the value of the constant a such that $x^3 - 7x^2 + (a+6)x - a = 0$ has a repeated root.

2 Find the value of the constant a such that $x^3 - (a+1)x^2 + (3a+5)x - 2a - 5 = 0$ has a repeated root.

3 Find the value of the constant a such that $x^3 + (2a-1)x^2 - 3(a-2)x + a - 6 = 0$ has a repeated root.

★ Challenge

Find the value of the constant a such that $x^3 + (a-1)x^2 + (a-3)x - 2a + 3 = 0$ has a repeated root.

Reflect

A quadratic equation (like $x^2 + 1 = 0$)might have no real solutions. Why does a cubic equation (like $x^3+x+1=0$) always have at least one real solution? What does this show about the way these two types of equations behave?

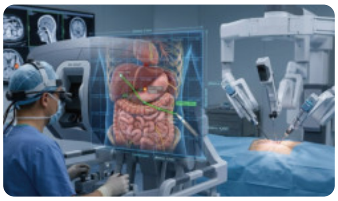
4-1

MATHEMATICS

Internal and External Division on a Number Line

Chapter 4 Geometry and Equations

Concept 1 · Coordinate Geometry Basics



When a surgeon plans a minimally invasive robotic surgery, precision is a matter of life and death. The robotic arm must navigate through a three-dimensional space inside the human body to target a tumor without damaging surrounding healthy tissue, blood vessels, or critical nerve endings.

To achieve this, the medical imaging software maps the patient's anatomy onto a precise grid network. Every anatomical landmark, incision site, and surgical tool tip is assigned an exact address of numerical coordinates. By tracking these mathematical positions, the surgical system determines the optimal paths, distances, and boundary midpoints required to execute the procedure safely.

Guiding Question: How does mapping physical space onto a grid of numbers allow us to precisely partition paths, find balances, and safely navigate complex physical environments?

Learning Outcomes

Find the coordinates of the division point (internal and external) of a line segment on the number line.

Point!

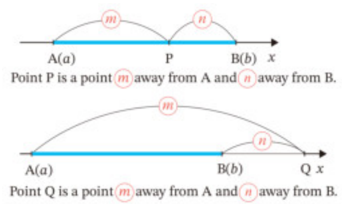
! On a number line, one real number a corresponds to point P. In this case, a is called the coordinate of point P, and point P with coordinate a is denoted as P(a).

! The distance AB between two points A(a) and B(b) on a number line is given by AB $=$ $|b-a|$

Since distance cannot be negative, we use the absolute value.

! If point P is on the line segment AB such that AP:PB $= m : n$, point P is said to divide the line segment AB **internally** in the ratio $m : n$.

! If point Q is on the extension of the line segment AB such that AQ:QB $= m : n$, point Q is said to divide the line segment AB **externally** in the ratio $m : n$.



! Regarding the line segment AB connecting two points A(a) and B(b) on a number line:

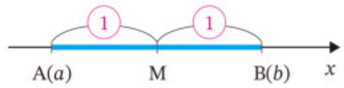
(i) The coordinate of point P dividing segment AB internally in the ratio $m : n$ is P$\left(\dfrac{na+mb}{m+n}\right)$

(ii) The coordinate of point Q dividing segment AB externally in the ratio $m : n$ is Q$\left(\dfrac{-na+mb}{m-n}\right)$

$\dfrac{na+mb}{m+n}$ $\dfrac{-na+mb}{m-n}$

Be careful with the order of multiplication (cross-multiply).

(iii) The coordinate of the midpoint M is (since it divides segment AB internally in 1:1): M $\left(\dfrac{a+b}{2}\right)$



Warm Up

For two points A(− 4) and B(8), find the following distances or coordinates.

(a) Distance AB between the two points.

(b) Coordinate of point C dividing segment AB internally in the ratio 2:1

(c) Coordinate of point D dividing segment AB externally in the ratio 2:3

(d) Coordinate of midpoint M of segment AB.

Explanation

(a) $AB = |8 - (-4)|$

$= 12$

(b) $\dfrac{1 \times (-4) + 2 \times 8}{2+1} = 4$ Therefore, C(4)

(c) $\dfrac{-3 \times (-4) + 2 \times 8}{2-3} = -28$ Therefore, D(− 28)

(d) $\dfrac{-4+8}{2} = 2$ Therefore, M(2)

Try

For two points A(− 5) and B(7), find the following distances or coordinates.

(a) Distance AB between the two points.

(b) Coordinate of point C dividing segment AB internally in the ratio 2:1

(c) Coordinate of point D dividing segment AB externally in the ratio 2:5

(d) Coordinate of midpoint M of segment AB.

> **Work with a partner**
>
> Collaborate with your group to solve parts (c and d) then discuss the answer.

Exercise

Solve the following problems.

1 For two points A(− 2) and B(6), find the following distances or coordinates.

(a) Distance AB between the two points.

(b) Coordinate of point C dividing segment AB internally in the ratio 3:1

(c) Coordinate of point D dividing segment AB externally in the ratio 5:3

(d) Coordinate of midpoint M of segment AB.

2 For two points A(4) and B(10), find the following distances or coordinates.

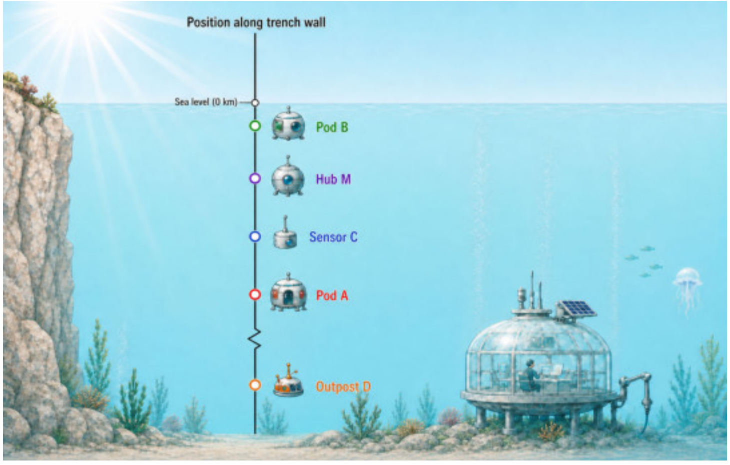
(a) Distance AB between the two points.

(b) Coordinate of point C dividing segment AB internally in the ratio 2:1

(c) Coordinate of point D dividing segment AB externally in the ratio 1:3

(d) Coordinate of midpoint M of segment AB.

★ Challenge



Subterranean Research Lab Infrastructure

A team of marine biologists is operating an underwater research habitat anchored along a straight vertical trench wall. To track positions along this deep-sea wall, they use a 1D reference axis measured in kilometers relative to sea level (0 km).

Currently, two primary monitoring pods are anchored directly to the trench wall:

Pod A is positioned at a depth of −8 km.

Pod B is positioned closer to the surface at a depth of −3 km.

Using your knowledge of coordinate division, calculate the exact locations and spacing required to expand their research network:

(a) Fiber-Optic Tether Length

(b) Internal Environmental Sensor Array — Point C divides AB internally in ratio 1:4

(c) External Deep-Trench Outpost — Point D divides AB externally in ratio 3:4

(d) Emergency Supply Hub Location (M) — Midpoint M of AB

4-2

MATHEMATICS

Points on the Coordinate Plane

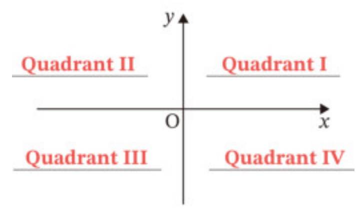
Chapter 4 Geometry and Equations

Point!

❗ The coordinate plane is divided into 4 quadrants as shown in the figure on the right, excluding the coordinate axes.

Learning Outcomes

Determine the quadrant of a point on the coordinate plane.



Warm Up

Solve the following problems.

1 Determine which quadrant the following points lie in.

(a) Point A$(2, 3)$ **(b)** Point B$(2, -3)$ **(c)** Point C$(-2, 3)$

(d) Point D$(-2, -3)$

2 For point A$(-3, 2)$, find the coordinates of the following points.

(a) Point P symmetric with respect to the x—axis

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

Explanation

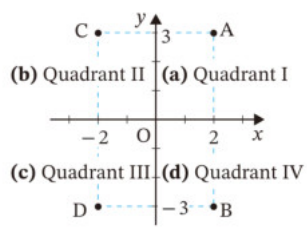
1 Plot the points on the coordinate plane to visualize.

(a) 1st Quadrant

(b) 4th Quadrant

(c) 2nd Quadrant

(d) 3rd Quadrant



2 Change the sign of other coordinate.

(a) Point P symmetric with respect to the x—axis

Change the sign of the y—coordinate.

<u>P$(-3, -2)$</u>

Change the sign of the x—coordinate.

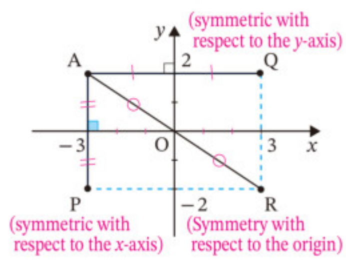
<u>Q$(3, 2)$</u>

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

Change the sign of the x—coordinate and the y—coordinate.

<u>R$(3, -2)$</u>



Try

Solve the following problems.

1 Determine which quadrant the following points lie in.

(a) Point A$(-4, 2)$ **(b)** Point B$(1, 2)$ **(c)** Point C$(3, -2)$

(d) Point D$(-3, -1)$

2 For point A$(-1, 2)$, find the coordinates of the following points.

(a) Point P symmetric with respect to the x—axis

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Determine which quadrant the following points lie in.

(a) Point A$(-3, 1)$ **(b)** Point B$(2, -8)$ **(c)** Point C$(4, 4)$

(d) Point D$(-6, -4)$ **(e)** Point E$(-2, 6)$ **(f)** Point F$(3, -3)$

(g) Point G$(-5, -7)$ **(h)** Point H$(1, 4)$

2 For point A$(-4, -2)$, find the coordinates of the following points.

(a) Point P symmetric with respect to the x—axis

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

3 For point A$(1, -2)$, find the coordinates of the following points.

(a) Point P symmetric with respect to the x—axis

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

4 For point A$(-3, 4)$, find the coordinates of the following points.

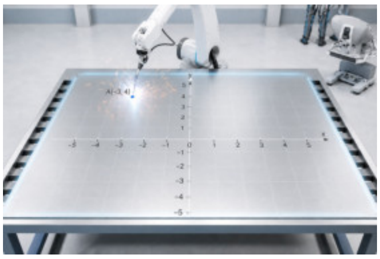
(a) Point P symmetric with respect to the x—axis

(b) Point Q symmetric with respect to the y—axis

(c) Point R symmetric with respect to the origin

★ Challenge

A robotic welding arm is programmed at point A(−3, 4) on a metal sheet. To weld identical support brackets, the arm must mirror this point:



(a) Point P symmetric with respect to the x-axis

(b) Point Q symmetric with respect to the y-axis

(c) Point R symmetric with respect to the origin

4-3
MATHEMATICS

Distance Between Two Points

Chapter 4 Geometry and Equations

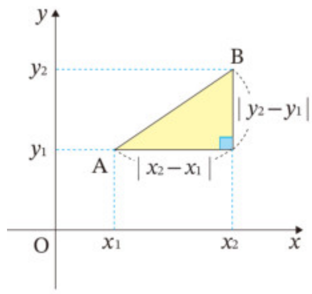
Point!

! The distance AB between two points A(x_1, y_1) and B(x_2, y_2) is
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

! In problems where the distance between 2 points is known,
square both sides .

Learning Outcomes

Calculate the distance between two points.



Use the Pythagorean theorem.

Warm Up

Solve the following problems.

1 Find the distance between the following 2 points. A(2, 6), B(5, 10)

2 If the distance between points A(3, − 2) and B(x, 1) is 5, find the value of x.

Explanation

1 $\sqrt{(5-2)^2 + (10-6)^2}$

$= \sqrt{3^2 + 4^2}$

$= 5$

2 Since the distance between 2 points A and B is 5,

$\sqrt{(x-3)^2 + \{1-(-2)\}^2} = 5$

Square both sides.

$(x-3)^2 + \{1-(-2)\}^2 = 25$

Solving this, $x = \underline{-1,\ 7}$

Try

Solve the following problems.

1 Find the distance between the following 2 points.

(a) A(4, 3), B(1, 2) **(b)** A($\sqrt{3}$, −1), B(0, 1) **(c)** Origin O, A(2, − 4)

2 If the distance between points A(−1, 3) and B(x, 1) is $\sqrt{5}$, find the value of x.

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Find the distance between the following 2 points.

(a) A(1, 2), B(11, 7) **(b)** A(3, −1), B(−1, 3) **(c)** Origin O, A(− 3, 4)

2 If the distance between points A(3, − 1) and B(x, 2) is 5, find x.

3 If the distance between points A(1, −3) and B(−2, y) is $\sqrt{13}$, find y.

4 If the distance between points A(−4, −2) and B(x, 3) is $5\sqrt{5}$, find x.

★ Challenge

If the distance between the two points (x,7) and (3x-1, -5) equals 13 length units, then find the value(s) of x.

4-4
MATHEMATICS

Points Equidistant from Two Points

Chapter 4 Geometry and Equations

Point!

! To find a point equidistant from two points, assign variables to its coordinates based on the given conditions.

Point on x—axis $\Longrightarrow$ The required point is (s, 0)

Point on y—axis $\Longrightarrow$ The required point is (0, t)

Point on line $y = ax + b$ $\Longrightarrow$ The required point is (s, as+b)

Learning Outcomes

Determine the position of a point equidistant from two points.

Warm Up

Solve the following problems.

1 Find the coordinates of point P on the x-axis equidistant from 2 points A(−2, 1) and B(3, 4).

2 Find the coordinates of point Q on $y = 2x - 1$ equidistant from 2 points A(1, 4) and B(3, 1).

Explanation

1 Let the coordinates of point P be $(s,\ 0)$.

Since it is equidistant from 2 points A(−2, 1) and B(3, 4),

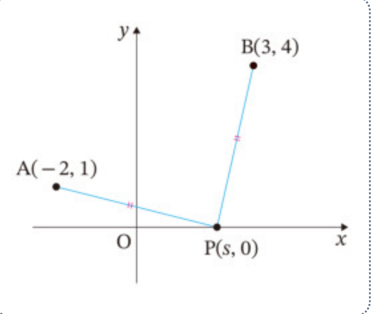
$\mathrm{AP} = \mathrm{BP}$

$\sqrt{\{s-(-2)\}^2+(-1)^2} = \sqrt{(s-3)^2+(-4)^2}$

$\{s-(-2)\}^2+(-1)^2 = (s-3)^2+(-4)^2$

Simplifying and solving this gives $s = 2$

Therefore, the required coordinates is $\underline{\mathrm{P}(2,\ 0)}$.



2 Let the coordinates of point Q be $(s,\ 2s-1)$.

Since it is equidistant from 2 points A(1, 4) and B(3, 1),

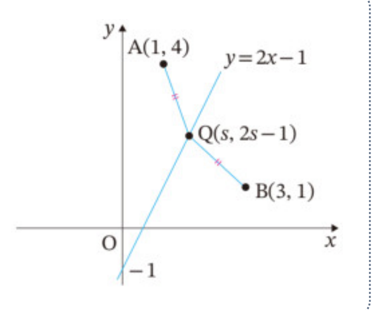
$\mathrm{AQ} = \mathrm{BQ}$

$\sqrt{(s-1)^2+\{(2s-1)-4\}^2} = \sqrt{(s-3)^2+\{(2s-1)-1\}^2}$

$(s-1)^2+\{(2s-1)-4\}^2 = (s-3)^2+\{(2s-1)-1\}^2$

Simplifying and solving this gives $s = \dfrac{13}{8}$

Since the required point is on $y = 2x-1$, $\underline{\mathrm{Q}\left(\dfrac{13}{8},\ \dfrac{9}{4}\right)}$



Try

Solve the following problems.

1 Find the coordinates of point P on the y-axis equidistant from 2 points A$(4, 2)$ and B$(1, -1)$.

2 Find the coordinates of point Q on $y = 3x + 1$ equidistant from 2 points A$(4, -5)$ and B$(2, -3)$.

Work with a partner

Work with your colleagues

Exercise

Solve the following problems.

1 Find the coordinates of point P on the x-axis equidistant from 2 points A$(-1, 2)$ and B$(4, 3)$.

2 Find the coordinates of point P on the y-axis equidistant from 2 points A$(-5, 2)$ and B$(3, 5)$.

3 Find the coordinates of point P on the x-axis equidistant from 2 points A$(-1, 2)$ and B$(3, 8)$.

4 Find the coordinates of point Q on $y = 2x - 3$ equidistant from 2 points A$(1, 4)$ and B$(3, -2)$.

5 Find the coordinates of point Q on $y = x - 1$ equidistant from 2 points A$(-1, 5)$ and B$(7, -1)$.

6 Find the coordinates of point Q on $y = -x + 1$ equidistant from 2 points A$(1, 2)$ and B$(-3, 6)$.

★ Challenge

If A (2,1), B (4,5), C (3, y) and D (x,0) such that CA = CB and DA = DB, then find CD.

4-5 MATHEMATICS

Internal and External Division Points on the Coordinate Plane

Chapter 4 Geometry and Equations

Point!

Regarding the line segment AB connecting two points A(x_1, y_1) and B(x_2, y_2) on the coordinate plane:

(a) The coordinates of point P dividing segment AB internally in the ratio $m:n$ are $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$

(b) The coordinates of point Q dividing segment AB externally in the ratio $m:n$ are $\left(\frac{-nx_1+mx_2}{m-n}, \frac{-ny_1+my_2}{m-n}\right)$

(c) Since the midpoint M divides segment AB internally in the ratio 1:1, its coordinates are M $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

The coordinates of the centroid G of △ABC with vertices A(x_1, y_1), B(x_2, y_2), and C(x_3, y_3) are:

G $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Learning Outcomes

Find the coordinates of the division point (internal and external) of a line segment on the coordinate plane.

$\frac{na+mb}{m+n}$ $\frac{-na+mb}{m-n}$

Pay attention to the order of multiplication.

Warm Up

Given three points A(− 2, − 3), B(3, 7), and C(5, 2), find the coordinates of the following points:

(a) Point P dividing segment BA internally in the ratio 1:2

(b) Point Q dividing segment BC externally in the ratio 1:3

(c) Midpoint M of segment CA

(d) Centroid G of triangle ABC

Explanation

(a) $\left(\frac{2\times3+1\times(-2)}{1+2}, \frac{2\times7+1\times(-3)}{1+2}\right)$, so P $\left(\frac{4}{3}, \frac{11}{3}\right)$

(b) $\left(\frac{-3\times3+1\times5}{1-3}, \frac{-3\times7+1\times2}{1-3}\right)$, so Q $\left(2, \frac{19}{2}\right)$

(c) $\left(\frac{5+(-2)}{2}, \frac{2+(-3)}{2}\right)$, so M $\left(\frac{3}{2}, -\frac{1}{2}\right)$

(d) $\left(\frac{-2+3+5}{3}, \frac{-3+7+2}{3}\right)$, so G(2, 2)

Try

Given three points $A(-3, 4)$, $B(1, -4)$, and $C(-4, 3)$, find the coordinates of the following points:

1. Point P dividing segment AC internally in the ratio 3:1
2. Point Q dividing segment BC externally in the ratio 2:3
3. Midpoint M of segment AB
4. Centroid G of triangle ABC

Work with a partner

Collaborate with your group to solve question (3 and 4) then discuss the answer.

Exercise

Solve the following problems.

1 Given three points $A(2, 3)$, $B(2, -3)$, and $C(-2, 3)$, find the coordinates of the following points:

(a) Point P dividing segment AB internally in the ratio 4:1
(b) Point Q dividing segment BC externally in the ratio 2:3
(c) Midpoint M of segment CA
(d) Centroid G of triangle ABC

2 Given three points $A(-2, 1)$, $B(6, -3)$, and $C(1, 7)$, find the coordinates of the following points:

(a) Point P dividing segment BC internally in the ratio 3:2
(b) Point Q dividing segment CA externally in the ratio 3:2
(c) Midpoint M of segment AB
(d) Centroid G of triangle ABC

3 Given three points $A(-4, 5)$, $B(1, -5)$, and $C(3, 0)$, find the coordinates of the following points:

(a) Point P dividing segment BA internally in the ratio 1:2
(b) Point Q dividing segment BC externally in the ratio 1:3
(c) Midpoint M of segment BC
(d) Centroid G of triangle ABC

★ Challenge

If $A(-2,3)$, $B(6,1)$ and $C(0, y)$, such that $CA = CB$, then find the coordinates of the centroid G of triangle ABC.

4-6

MATHEMATICS

Symmetric Points

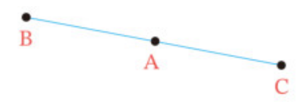
Chapter 4 Geometry and Equations

Point!

! Two points B, C are symmetric with respect to point A ⟺ Point A is the midpoint of segment BC. When finding symmetric points, first draw a diagram to understand the relative positions of the points. Next, let the required coordinates be $(s,\ t)$ and **express the midpoint in terms of s, t**.

Learning Outcomes

Determine the symmetric point.



Warm Up

Find the coordinates of point B symmetric to point A$(-3, 2)$ with respect to point P$(2, 6)$

Explanation

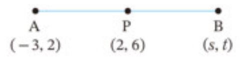
Let the coordinates of point B be $(s,\ t)$.

The coordinates of the midpoint of segment AB can be expressed as

$$\left(\frac{-3+s}{2},\ \frac{2+t}{2}\right)$$

Express the midpoint in terms of $s,\ t.$

Since this coincides with point P$(2, 6)$,



$$\begin{cases} \dfrac{-3+s}{2} = 2 \\ \dfrac{2+t}{2} = 6 \end{cases}$$

Solving this gives $s = 7, t = 10$

Therefore, the required coordinates are $B(7,\ 10)$.

Try

Solve the following problems.

1 Find the coordinates of point B symmetric to point A$(1, -2)$ with respect to point P$(2, 3)$.

2 Find the coordinates of point Q symmetric to point P$(3, 2)$ with respect to point A$(-1, 5)$.

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Find the coordinates of point B symmetric to point $A(-3, 1)$ with respect to point $P(2, 0)$.

2 Find the coordinates of point B symmetric to point $A(7, 4)$ with respect to point $M(-3, 5)$.

3 Find the coordinates of point B symmetric to point $A(4, -8)$ with respect to point $P(-3, 2)$.

4 Find the coordinates of point Q symmetric to point $P(0, -4)$ with respect to point $A(-4, 1)$.

5 Find the coordinates of point B symmetric to point $A(-4, 2)$ with respect to point $P(-1, -3)$.

★ Challenge

If the points B $(m, 5)$, C $(4, n)$ are symmetric with respect to the point A $(1, -1)$, then find $m + n$.

Reflect

If we change the ratio of dividing a line segment, how will the position of the point on that line change? And how does thinking about ratios and reference distances help us understand locations in mathematics and in everyday life?

4-7

MATHEMATICS

Shapes of Triangles

Chapter 4 Geometry and Equations

Concept 2 · Triangles and Lines

In modern medicine, precisely locating a target inside the human body is a matter of life and death. During Stereotactic Radiosurgery, multiple narrow beams of radiation are aimed from different angles to intersect precisely at a tumor.

By modeling these radiation beams as straight lines in a three-dimensional coordinate system, medical physicists calculate their exact equations to ensure they intersect at a single point (the tumor) while minimizing damage to the surrounding healthy tissue. If the lines do not intersect correctly, or if they form an accidental triangle around the tumor instead of meeting exactly at a point, the treatment fails.

Guiding Question How can we use the algebraic equations of straight lines to guarantee that multiple paths intersect at a single, exact target point, and how do we determine the geometric boundaries of the areas they influence?

Learning Outcomes

Use coordinates to determine types of triangles (isosceles, right-angled, etc.).

Point!

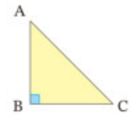
- To determine the shape of △ABC, calculate $(AB)^2$, $(BC)^2$, and $(CA)^2$.
- In △ABC:

If $(AB)^2 = (BC)^2 = (CA)^2$, △ABC is an **Equilateral triangle** .

If $(AB)^2 = (BC)^2 \neq (CA)^2$, △ABC is an **Isosceles triangle** .

If $(AC)^2 = (AB)^2 + (BC)^2$, △ABC is a **Right-angled triangle** .

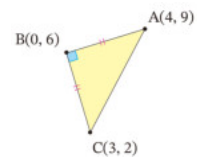
Note: If both conditions are met, it is an **Isosceles right-angled triangle** .



$(AC)^2=(AB)^2+(BC)^2$

Warm Up

Determine the shape of △ABC with vertices A(4, 9), B(0, 6), and C(3, 2).



Explanation

$(AB)^2 = (0-4)^2 + (6-9)^2$

$\quad = 25 \ldots\ldots$ (i)

$(BC)^2 = (3-0)^2 + (2-6)^2$

$\quad = 25 \ldots\ldots$ (ii)

$(CA)^2 = (3-4)^2 + (2-9)^2$

$\quad = 50 \ldots\ldots$ (iii)

From(i)and (ii), △ABC is an isosceles triangle.

From(i), (ii), and (iii), since $(CA)^2 = (AB)^2 + (BC)^2$, △ABC is a right-angled triangle.

Therefore, △ABC is an Isosceles right-angled triangle with ∠B = 90°.

Try

Solve the following problems.

1 Determine the shape of △ABC with vertices A(− 1, 2), B(1, 1), and C(0, 4).

2 Determine the shape of △ABC with vertices A(3, 2), B(−1, 0), and C($1+\sqrt{3}$, $1-2\sqrt{3}$).

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Determine the shape of △ABC with vertices A(0, 2), B(− 1, − 2), and C(4, 1).

2 Determine the shape of △ABC with vertices A(2, 2), B(2, 0), and C($\sqrt{3}+2$, 1).

3 Determine the shape of △ABC with vertices A(1, 2), B(5, 8), and C(11, 4).

4 Determine the shape of △ABC with vertices A(− 3, 4), B(1, 7), and C(3, − 4).

5 Determine the shape of △ABC with vertices A(− 1, 1), B(4, 2), and C(1, 4).

★ Challenge

If A (1,2), B (2k, k) and C (4,5) and the triangle ABC is an isosceles and right triangle at B, then find k.

4-8

MATHEMATICS

Equation of a Straight Line

Chapter 4 Geometry and Equations

Learning Outcomes

Find the equation of the straight line in its different forms.

Point!

! Equation of a line parallel to an axis

- The equation of a line passing through point (x_1, y_1) and **parallel to the x–axis** is $y = y_1$
- The equation of a line passing through point (x_1, y_1) and **perpendicular to the x–axis** is $x = x_1$
- The equation of a line passing through point (x_1, y_1) and **parallel to the y–axis** is $x = x_1$
- The equation of a line passing through 2 points (x_1, y_1) and (x_2, y_1) is $y = y_1$

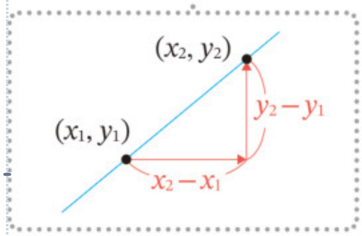
The y-coordinates of the 2 points are equal.

- The equation of a line passing through 2 points (x_1, y_1) and (x_1, y_2) is $x = x_1$

The x-coordinates of the 2 points are equal.

! If the gradient and a point on the line are known, the equation of the line can be determined.

- The equation of a line passing through point (x_1, y_1) with gradient a is $y - y_1 = a(x - x_1)$.
- The gradient of a line passing through 2 points (x_1, y_1) and $(x_2, y_2)(x_1 \neq x_2)$ is $a = \dfrac{y_2 - y_1}{x_2 - x_1}$



Warm Up

Find the equations of the following lines.

(a) The line passing through point $(3, -4)$ and parallel to the x-axis

(b) The line passing through point $(-3, 8)$ and parallel to the y-axis

(c) The line passing through two points $(-3, 1)$ and $(-3, 2)$

(d) The line passing through two points $(-2, 3)$ and $(-5, 3)$

(e) The line passing through point $(2, -1)$ with gradient 3

(f) The line passing through two points $(1, -1)$ and $(4, 0)$

Explanation

(*a*) $y = -4$

(*b*) $x = -3$

(*c*) Since the x-coordinates of the 2 points are equal,

$x = -3$

(*d*) Since the y-coordinates of the 2 points are equal,

$y = 3$

(*e*) $y - (-1) = 3\,(x-2)$

$y = 3x-7$

(*f*) $y-(-1) = \dfrac{0-(-1)}{4-1}(x-1)$

$y = \dfrac{1}{3}x - \dfrac{4}{3}$

Express the equation in the form $y = mx + k$.

$a = \dfrac{y_2 - y_1}{x_2 - x_1}$

Try

Find the equations of the following lines.

1 The line passing through point (2, 6) and parallel to the x-axis

2 The line passing through point (3, 5) and perpendicular to the x-axis

3 The line passing through two points (− 2, − 2) and (2, − 2)

4 The line passing through two points (− 4, 2) and (− 4, 8)

5 The line passing through point (− 3, 4) with gradient 2

6 The line passing through two points (− 2, − 6) and (− 4, 1)

Work with a partner

Collaborate with your group to solve questions (5 and 6) then discuss the answer.

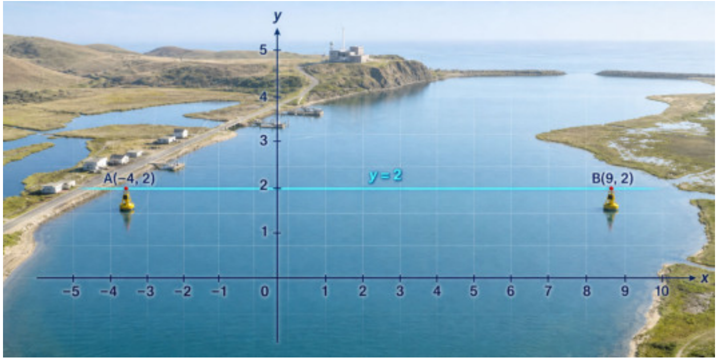
Exercise

Solve the following problems.

1 Find the equations of the following lines.

(a) The line passing through point $(-4, 5)$ and parallel to the x-axis
(b) The line passing through point $(2, 1)$ and parallel to the x-axis
(c) The line passing through point $(-4, -7)$ and parallel to the x-axis
(d) The line passing through point $(1, -2)$ and perpendicular to the x-axis
(e) The line passing through point $(-3, -4)$ and perpendicular to the x-axis
(f) The line passing through point $(3, -5)$ and perpendicular to the x-axis
(g) The line passing through point $(2, -7)$ and parallel to the y-axis
(h) The line passing through point $(4, 3)$ and parallel to the y-axis
(i) The line passing through point $(-6, -2)$ and parallel to the y-axis
(j) The line passing through two points $(4, 0)$ and $(4, 3)$
(k) The line passing through two points $(-1, 2)$ and $(-1, 6)$
(l) The line passing through two points $(-5, -3)$ and $(-5, 2)$
(m) The line passing through two points $(-2, 3)$ and $(-3, 3)$
(n) The line passing through two points $(4, -2)$ and $(1, -2)$
(o) The line passing through two points $(-3, 6)$ and $(2, 6)$
(p) The line passing through point $(-1, 5)$ with gradient -1
(q) The line passing through point $(-3, -8)$ with gradient 1
(r) The line passing through point $(2, -1)$ with gradient -2
(s) The line passing through two points $(1, 2)$ and $(3, 10)$
(t) The line passing through two points $(-1, -8)$ and $(4, 2)$

★ Challenge



An environmental research team is monitoring sea-level changes in a sensitive coastal estuary. They have set up a grid system where:

The x-axis represents the distance from a central research station

The y-axis represents the local sea-surface height

During a tidal study, sensors at two different monitoring stations record identical, stable sea-surface heights:

Sensor West is located at coordinates A(-4, 2).

Sensor East is located at coordinates B(9, 2).

To map the uniform boundary of this water level across the estuary, the team needs to find the equation of the straight-line boundary connecting these two sensors.

4-9

MATHEMATICS

Application of the Equation of a Straight Line

Chapter 4 Geometry and Equations

Point!

! An equation of the form $ax + by + c= 0$(where $a \neq 0$ or $b \neq 0$) also represents a straight line.

! The equation of a straight line can be found when the gradient and a point on the line are known.

- The gradient of a line can be found using two points on the line.

Learning Outcomes

The equation of the straight line in its different forms, and its application in geometric situations.

Warm Up

Solve the following problems.

1 Given that three points A(2, 3), B(4, − 1), and C(a − 1, 2 − a) lie on the same straight line, find the value of the constant a.

2 Find the equation of the line passing through the intersection of 2 lines $2x - 3y + 1 = 0$ and $x + 2y - 3 = 0$, and passing through point (− 1, 3).

Explanation

1 The equation of the line passing through 2 points A(2, 3) and B(4, − 1) is:

$$y - 3 = \frac{-1-3}{4-2}(x-2)$$

Simplifying this gives $y = -2x + 7$.. … .(i)

Since (i) passes through C($a - 1,\ 2 - a$) :

$$2 - a = -2(a-1) + 7$$

Solving this gives $a = \underline{7}$

Find the equation of the line from 2 points whose coordinates are known.

Substitute the coordinates of point C into (i).

2 $2x - 3y + 1 = 0$ …… (i)

$x + 2y - 3 = 0$ …… (ii)

Solving this system of equations gives $x = 1, y = 1$

The equation of the line passing through 2 points (1, 1) and (− 1, 3) is:

$$y - 1 = \frac{3-1}{-1-1}(x-1)$$

That is, $\underline{x + y - 2 = 0}$

Solve the equations of the 2 lines simultaneously to find the point of intersection.

The coordinates of the intersection of (i) and (ii) are (1, 1).

Answer in the form $ax + by + c = 0$ as in the question.

Try

Solve the following problems.

1 Given that three points A$(2, 5)$, B$(4, 9)$, and C$(-1, a)$ lie on the same straight line, find the value of the constant a.

2 Find the equation of the line passing through the intersection of 2 lines $x + 2y - 10 = 0$ and $2x + 3y - 7 = 0$, and passing through point $(5, 6)$.

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Given that three points A$(1, -1)$, B$(-2, -3)$, and C$(4, a)$ lie on the same straight line, find the value of the constant a.

2 Given that three points A$(1, 2)$, B$(3, 1)$, and C$(a, -1)$ lie on the same straight line, find the value of the constant a.

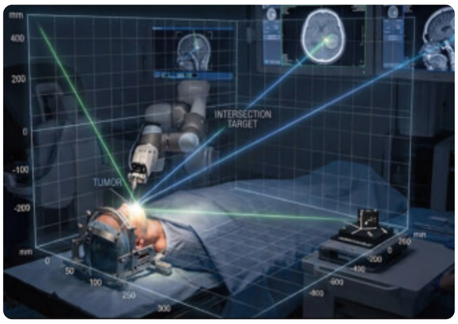
3 Given that three points A$(-3, -9)$, B$(5, 7)$, and C$(a + 3, 3a - 2)$ lie on the same straight line, find the value of the constant a.

4 Find the equation of the line passing through the intersection of 2 lines $3x + 5y - 1 = 0$ and $7x + 8y - 6 = 0$, and passing through point $(-1, 2)$.

5 Find the equation of the line passing through the intersection of 2 lines $x + 2y - 4 = 0$ and $2x - y - 3 = 0$, and passing through point $(-3, -1)$.

★ Challenge

Robotic Needle Biopsy Guidance



In robotic-assisted surgery, a computer system guides a biopsy needle to a deep-seated tumor. The target tumor is located at the precise intersection of two diagnostic imaging paths (modeled as straight lines on a coordinate grid, where 1 unit = 1 millimeter):

- Laser Scan Path A: $6x + 5y + 7 = 0$
- Laser Scan Path B: $3x - 2y + 5 = 0$

The Clinical Challenge:

The surgical robot's mechanical arm is anchored at the system's coordinate origin (0,0) (the home calibration point). To perform the biopsy, the robotic arm must insert a rigid needle along a perfectly straight trajectory from the origin (0,0) directly to the tumor at the intersection point.

The software engineers must calculate the exact linear equation of this needle trajectory to program the robotic arm's path.

4-10

MATHEMATICS

Parallel and Perpendicular Lines

Chapter 4 Geometry and Equations

Point!

! An equation of the form $ax + by + c = 0$ (where $a \neq 0$ or $b \neq 0$) also represents a straight line.

! Problems concerning parallel or perpendicular lines use gradients.

For 2 lines $y = m_1x + k_1$ and $y = m_2x + k_2$:

2 lines are parallel $\Longrightarrow m_1 = m_2$, that is, gradients are equal.

2 lines are perpendicular $\Longrightarrow m_1 = -\dfrac{1}{m_2}$, that is,

gradients are negative reciprocals

! Solve problems concerning parallel or perpendicular lines by transforming the equation into the form $\boldsymbol{y = mx + k}$.

Learning Outcomes

Apply the conditions for parallelism and perpendicularity of straight lines to solve geometric problems.

Warm Up

Solve the following problems.

1 Identify which of the following lines are parallel to each other and which are perpendicular.

(a) $y = 2x + 3$ **(b)** $y = -3x + 2$ **(c)** $x - 3y + 1 = 0$

(d) $2x - y - 1 = 0$ **(e)** $x + 3y + 3 = 0$

2 Find the equation of the line passing through point $(2, -4)$ and parallel to $3x - 2y + 1 = 0$

3 Find the equation of the line passing through point $(-2, 5)$ and perpendicular to $3x + 5y + 1 = 0$

Explanation

1 Find the gradient of each line.

Transform the equation into the form $y = mx + k$.

(a) $y = 2x + 3$, the gradient is 2

(b) $y = -3x + 2$, the gradient is -3

(c) $x - 3y + 1 = 0$

$y = \frac{1}{3}x + \frac{1}{3}$, the gradient is $\frac{1}{3}$

(d) $2x - y - 1 = 0$

$y = 2x - 1$, the gradient is 2

(e) $x + 3y + 3 = 0$

$y = -\frac{1}{3}x - 1$, the gradient is $-\frac{1}{3}$

Parallel lines are **(a)** and **(d)** .

Choose those with equal gradients.

Perpendicular lines are **(b)** and **(c)** .

Choose those whose gradients are negative reciprocals.

2 $3x - 2y + 1 = 0$

Transform into the form $y = mx + k$ and find the gradient.

$y = \frac{3}{2}x + \frac{1}{2}$, the gradient is $\frac{3}{2}$

Therefore, the required parallel line passes through point $(2, -4)$ and has a gradient of $\frac{3}{2}$

The gradient of a parallel line is the same.

$y - (-4) = \frac{3}{2}(x - 2)$

Simplifying this gives

$\underline{3x - 2y - 14 = 0}$

Match the equation to the form in the question $(ax + by + c = 0)$.

3 $3x + 5y + 1 = 0$

Transform into the form $y = mx + k$ and find the gradient.

$y = -\frac{3}{5}x - \frac{1}{5}$, the gradient is $-\frac{3}{5}$

Therefore, the required perpendicular line passes through point $(-2, 5)$ and has a gradient of $\frac{5}{3}$

The gradient of a perpendicular line is the negative reciprocal.

$y - 5 = \frac{5}{3}\{x - (-2)\}$

Simplifying this gives $\underline{5x - 3y + 25 = 0}$

Match the equation to the form in the question.

Try

Solve the following problems.

1 Identify which of the following lines are parallel to each other and which are perpendicular.

(a) $y = 3x + 1$ **(b)** $y = 4x - 1$ **(c)** $y = -4x - 7$

(d) $6x - 2y = 5$ **(e)** $8x + 2y - 3 = 0$ **(f)** $x + 4y - 4 = 0$

2 Find the equation of the line passing through point $(1, -3)$ and parallel to $6x + 3y - 5 = 0$

3 Find the equation of the line passing through point $(2, 1)$ and perpendicular to $2x + 3y + 4 = 0$

Work with a partner

Collaborate with your group to solve question (1) then discuss the answer.

Exercise

Solve the following problems.

1 Identify which of the following lines are parallel to each other and which are perpendicular.

(a) $y = 3x - 1$ **(b)** $y = -3x + 4$ **(c)** $y = x - 3$

(d) $6x + 2y + 5 = 0$ **(e)** $2x - 2y - 1 = 0$ **(f)** $x + 3y - 2 = 0$

2 Identify which of the following lines are parallel to each other and which are perpendicular.

(a) $y = 2x - 1$ **(b)** $y = -x + 4$ **(c)** $y = \frac{3}{2}x + 6$

(d) $x + y + 3 = 0$ **(e)** $2x + 3y + 8 = 0$ **(f)** $3x + 6y - 1 = 0$

3 Find the equation of the line passing through point $(1, -2)$ and parallel to $2x - y + 5 = 0$

4 Find the equation of the line passing through point $(-1, 2)$ and parallel to $2x + 3y = 0$

5 Find the equation of the line passing through point $(3, -1)$ and parallel to $3x + 2y + 1 = 0$

6 Find the equation of the line passing through point $(2, -5)$ and perpendicular to $2x + y - 1 = 0$

7 Find the equation of the line passing through point $(-1, 3)$ and perpendicular to $5x - 2y + 3 = 0$

8 Find the equation of the line passing through point $(-2, -5)$ and perpendicular to $3x - 2y - 7 = 0$

★ Challenge

For the straight lines $L_1: 2x + 3y - 10 = 0$, $L_2: kx - 2y + 5 = 0$, $L_3: 4x + my - 1 = 0$, if L_1 is perpendicular to L_2 and L_1 is parallel to L_3, then find m/k.

4-11
MATHEMATICS

Perpendicular Bisector and Symmetric Points with Respect to a Line

Chapter 4 Geometry and Equations

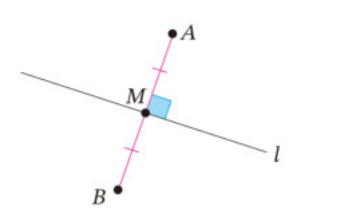
Point!

❗ Since the perpendicular bisector of line segment (AB) is perpendicular to line AB and passes through the midpoint of segment (AB), it is found by using the **gradient** of line (AB) and the **coordinates of the midpoint** of segment (AB).

❗ To find the coordinates of point B$(s,\ t)$, which is symmetric to point A with respect to line l, we express the following two conditions as equations and solve them simultaneously: **the line segment (AB) is perpendicular to l** and **the midpoint of (AB) lies on line l**

Learning Outcomes

Derive the equation of the perpendicular bisector of a line segment and identify the symmetry points of a straight line, applying these concepts in geometric contexts.



Warm Up

Solve the following problems.

1 Given points A$(5, 11)$ and B$(1, -1)$, find the equation of the perpendicular bisector of line segment (AB).

2 Find the coordinates of point B symmetric to point A$(-4, 3)$ with respect to line $3x - y + 5 = 0$

Explanation

1 The gradient of line (AB) is $\dfrac{-1-11}{1-5} = 3$, so the gradient of the perpendicular line is $-\dfrac{1}{3}$

The gradient of a perpendicular line is the negative reciprocal.

Also, the coordinates of the midpoint of segment (AB) are $\left(\dfrac{5+1}{2}, \dfrac{11-1}{2}\right)$, which is $(3, 5)$

Therefore, the perpendicular bisector of segment (AB) is a line passing through $(3, 5)$ with a gradient of $-\dfrac{1}{3}$, so:

$y-5 = -\dfrac{1}{3}(x-3)$. Simplifying this gives $x + 3y - 18 = 0$

2 Let the coordinates of point B be $(s,\ t)$, and the line $3x - y + 5 = 0$ be l.

The gradient of line (AB) is $\dfrac{t-3}{s+4}$, and the gradient of l is 3

Transform l into $y = 3x + 5$ to find the gradient.

Since $(AB) \perp l$, $\dfrac{t-3}{s+4} = -\dfrac{1}{3}$

The gradient of a perpendicular line is the negative reciprocal.

Simplifying this gives $s + 3t - 5 = 0$......(i)

Also, since the coordinates of the midpoint of segment (AB) $\left(\dfrac{-4+s}{2}, \dfrac{3+t}{2}\right)$ lie on l,

Substitute $x = \dfrac{-4+s}{2}$, $y = \dfrac{3+t}{2}$ into the equation of l

$3\frac{-4+s}{2} - \frac{3+t}{2} + 5 = 0$ Simplifying this gives $3s - t - 5 = 0$ ......(ii)

Solving (i) and (ii) simultaneously gives $s = 2, t = 1$ Therefore, the coordinates of point B are $\underline{B(2, 1)}$

Try

Solve the following problems.

1 Given points A(− 1, 2) and B(3, − 4), find the equation of the perpendicular bisector of line segment (AB).

2 Find the coordinates of point B symmetric to point A(0, 4) with respect to line $2x - y - 1 = 0$

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Solve the following problems.

1 Find the equations of the perpendicular bisectors of the following line segments AB.

(a) A(−2, 3), B(2, 1) **(b)** A(−1, 2), B(5, −2) **(c)** A(0, 6), B(4, 4)

2 Find the coordinates of point B symmetric to point A(1, 2) with respect to line $x + 2y - 10 = 0$

3 Find the coordinates of point B symmetric to point A(4, − 1) with respect to line $4x - 2y - 3 = 0$

★ Challenge

In an ophthalmic surgical laser system, a targeting laser line is modeled by the equation y = 2x. If a calibration sensor is placed at point A(5, 0), the signal receiver must be positioned at point B, which is the exact symmetric reflection of point A across the laser line.

Find the coordinates of target point B to ensure accurate system calibration.

4-12

MATHEMATICS

Distance Between a Point and a Line

Chapter 4 Geometry and Equations

Point!

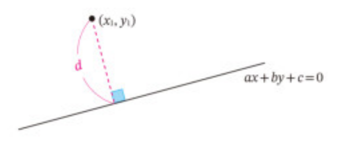
❗ The distance d between point (x_1, y_1) and line $ax + by + c = 0$ is:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

❗ To find the length of the perpendicular dropped from a point to a line, just find the distance between the point and the line.

Learning Outcomes

Calculate the perpendicular distance from a point to a straight line and apply this skill in solving geometric problems.



Warm Up

Find the following values.

1 The distance d between point $(-2, 1)$ and line $3x-2y+5=0$

2 The length of the perpendicular OH dropped from the origin O to line $3x - y = 5$

Explanation

1 $$d = \frac{|3\cdot(-2)+(-2)\cdot 1+5|}{\sqrt{3^2+(-2)^2}} = \frac{|-3|}{\sqrt{13}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

2 The length of the perpendicular dropped from a point to a line is the distance between the point and the line.

Transforming $3x - y = 5$ gives $3x - y - 5 = 0$

Transform into the form $ax + by + c = 0$ so the formula can be used.

$$OH = \frac{|3\cdot 0+(-1)\cdot 0-5|}{\sqrt{3^2+(-1)^2}} = \frac{\sqrt{10}}{2}$$

Try

Find the following values.

1. The distance between point $(3, -2)$ and line $x + 2y - 4 = 0$
2. The length of the perpendicular dropped from point $(-1, 5)$ to line $y = 3x - 2$

Work with a partner

Collaborate with your group to solve question (2) then discuss the answer.

Exercise

Find the following values.

1. The distance between point $(-3, 2)$ and line $2x - 3y + 6 = 0$
2. The distance between point $(-3, 7)$ and line $12x - 5y = 7$
3. The distance between point $(-8, 1)$ and line $y = 2x - 3$
4. The distance between the origin and line $3x - 4y - 5 = 0$
5. The length of the perpendicular dropped from point $(2, 3)$ to line $3x + 2y + 1 = 0$
6. The length of the perpendicular dropped from point $(13, 4)$ to line $3x + 4y - 5 = 0$
7. The length of the perpendicular dropped from point $(1, 2)$ to line $y = 3x + 1$

★ Challenge

In stereotactic radiosurgery, a narrow radiation beam is modeled by the line $y = \frac{1}{3}x + 2$. A critical nerve center (modeled as a point) is located nearby at $P(-1, 5)$.

To prevent radiation damage, medical physicists must calculate the shortest (perpendicular) distance from the nerve center to the path of the radiation beam. Find this distance to ensure it meets the required safety margin.

4-13

MATHEMATICS

Area of a Triangle

Chapter 4 Geometry and Equations

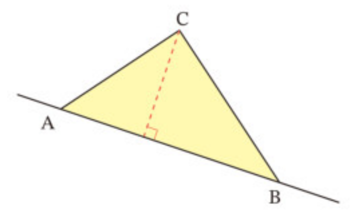
Point!

How to find the area of △ABC on the coordinate plane:

- For the base, find the **distance between 2 points A and B** .
- For the height, find the **distance between point C and line (AB)** .

Learning Outcomes

Determine the area of a triangle using coordinates.



Warm Up

Find the area of △ABC with vertices A(1, 1), B(3, 7), and C(−3, −1)

Explanation

$(AB) = \sqrt{(3-1)^2+(7-1)^2}$

Find the base.

$= 2\sqrt{10}$ ...... (i)

The equation of line (AB) is $y-1=\dfrac{7-1}{3-1}(x-1)$

Find the equation of line AB, which is needed for the height calculation.

Simplifying this gives $3x-y-2=0$

The distance between point C and line (AB) is:

Find the height.

$$\frac{|\,3\cdot(-3)+(-1)\cdot(-1)-2\,|}{\sqrt{3^2+(-1)^2}} = \sqrt{10} \text{ ...... } (ii)$$

From (i) and (ii), the required area of △ABC is:

$$\frac{1}{2}\times 2\sqrt{10}\times\sqrt{10} = 10$$

Try

Find the areas of △ABC given the following vertices.

(a) A$(1,\ 5)$, B$(-3,\ -3)$, C$(3,\ -6)$

(b) A$(-2,\ 1)$, B$(2,\ -1)$, C$(0,\ 4)$

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

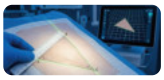
Find the areas of △ABC given the following vertices.

(a) A$(0,\ -6)$, B$(2,\ -2)$, C$(1,\ 5)$

(b) A$(-2,\ 4)$, B$(-3,\ -5)$, C$(5,\ -1)$

(c) A$(2,\ 5)$, B$(0,\ 1)$, C$(4,\ 2)$

★ Challenge



In reconstructive burn surgery, a patient requires a triangular skin graft. A surgical imaging system maps the wound boundaries onto a coordinate grid (where 1 unit = 1 mm) with vertices at A(-2, 1), B(1, 5), and C(-21, -21).

To harvest the correct amount of donor tissue, the surgical team must calculate the exact surface area of the triangular wound. Find the area of triangle ABC.

4-14

MATHEMATICS

Line Passing Through a Fixed Point

Chapter 4 Geometry and Equations

Point!

! Steps to solve problems involving lines “passing through a fixed point for all values of k” :

❶ **Rearrange the equation of the line with respect to k** and treat it as an **identity in k**.

❷ From the identity in k, $(ax+by+c)k+(a'x+b'y+c')=0$,

$$\begin{cases} ax+by+c=0 \\ a'x+b'y+c'=0 \end{cases}$$ create a system of equations and solve it.

Learning Outcomes

Determine the line passing through a fixed point.

Warm Up

The line $(k-4)x+(k+2)y+k+5=0$ passes through a fixed point regardless of k. Find the coordinates of that fixed point.

Explanation

Rearranging $(k-4)x+(k+2)y+k+5=0$ with respect to k gives:

$(x+y+1)k+(-4x+2y+5)=0$

Since this must hold for all k,

Treat it as an identity in k.

$$\begin{cases} x+y+1=0 \\ -4x+2y+5=0 \end{cases}$$

Solving this gives $x=\frac{1}{2},\ y=-\frac{3}{2}$

Therefore, the fixed point the line passes through is $\left(\frac{1}{2},\ -\frac{3}{2}\right)$

Try

The line $(2k+1)x-(k+2)y+7k+8=0$ passes through a fixed point for all values of k. Find the coordinates of this fixed point.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 The line $(k+2)x+ky-6=0$ passes through a fixed point for all values of k. Find the coordinates of this fixed point.

2 The line $(k+3)x-(2k+1)y+k-2=0$ passes through a fixed point for all values of k. Find the coordinates of this fixed point.

3 The line $(k+3)x+(3k+2)y+7=0$ passes through a fixed point for all values of k. Find the coordinates of this fixed point.

★ Challenge

The line $-(k-2)x+(k-3)y=-3k+5$ passes through a fixed point for all values of k. Find the coordinates of this fixed point.

4-15
MATHEMATICS

Conditions for Not Forming a Triangle

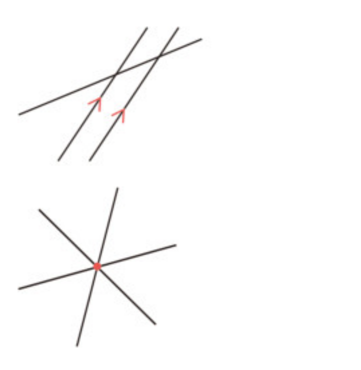
Chapter 4 Geometry and Equations

Point!

! There are 2 cases for the conditions where 3 lines do not form a triangle: ❶ **2 lines are parallel** $\Longrightarrow$ The gradients of the 2 lines are the same. ❷ **3 lines intersect at 1 point** $\Longrightarrow$ Find the point of intersection of 2 lines, and the remaining line also passes through that point.

Learning Outcomes

Derive and explain the conditions under which a triangle does not form (collinearity of points).



Warm Up

If the 3 lines $x - y + 1 = 0$, $2x + y + 2 = 0$, and $ax + 3y - 4 = 0$ do not form a triangle, find the value of the constant a.

Explanation

First, check the condition where 2 of the given 3 lines are parallel.

Transforming the 3 lines gives:

$$\begin{cases} y = x + 1 \text{ ...... (i)} \\ y = -2x - 2 \text{ ...... (ii)} \\ y = -\dfrac{a}{3}x + \dfrac{4}{3} \text{ ...... (iii)} \end{cases}$$

Transform into $y = mx + k$ form to see the gradients.

Since (i) and (ii) are not parallel, check the cases where (i) and (iii) are parallel, and (ii) and (iii) are parallel.

(i) and (iii) are parallel when:

The gradients of the 2 lines are the same.

$1 = -\dfrac{a}{3}$, so $a = -3$

(ii) and (iii) are parallel when: $-2 = -\dfrac{a}{3}$, so $a = 6$

Next, check the condition where the 3 lines intersect at 1 point.

Find the point of intersection of the 2 lines that do not have the constant a.

Solving (i) and (ii) simultaneously, the point of intersection is $(-1, 0)$.

The remaining line passes through this point.

Since (iii) passes through point $(-1, 0)$, $0 = -\dfrac{a}{3} \cdot (-1) + \dfrac{4}{3}$, so $a = -4$

Therefore, $\underline{a = -4,\ -3,\ 6}$

Try

If the 3 lines $x - 2y + 8 = 0$, $x + y - 1 = 0$, and $ax + y - 5 = 0$ do not form a triangle, find the value of the constant a.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1 If the 3 lines $x - 2y + 1 = 0$, $3x + 2y = 6$, and $ax - 3y + 2 = 0$ do not form a triangle, find the value of the constant a.

2 If the 3 lines $x - y = -1$, $3x + 2y = 12$, and $ax - y = a - 1$ do not form a triangle, find the value of the constant a.

★ Challenge

If the three lines $x + 2y - 1 = 0$, $3x + y + 2 = 0$ and $mx + 4y - 3 = 0$ do not form a triangle , then find the sum of all possible values of m.

Reflect

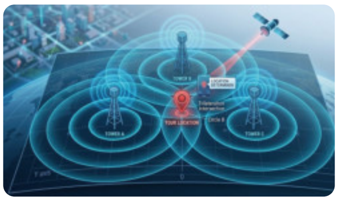
How might the intersection of lines and triangles be used to solve problems (e.g., navigation, construction, or design)?

4-16
MATHEMATICS

Equation of a Circle (1)

Chapter 4 Geometry and Equations

Concept 3 · Circles



Global positioning systems (GPS) and localized cellular networks rely entirely on the coordinate geometry of circles to determine your exact location. When your phone searches for a signal, it connects to multiple stationary cell towers. Each tower detects your device's signal strength and calculates a precise radius of distance, creating a circular boundary of where you could be.

A single tower leaves your location ambiguous, but the moment a second and third tower map their overlapping circular signals, coordinate geometry pinpoints your exact position. In the tech industry, this mathematical process is known as trilateration. A minor mathematical error in calculating these circular boundaries would disrupt everything from automated ride-sharing apps to emergency response tracking systems.

Guiding Question How can we use the coordinate geometry of circles to model boundaries, calculate overlapping regions, and determine the exact intersection points of wireless signals in modern communication systems?

Learning Outcomes

Derive and use the different forms of the equation of a circle—given its center and radius, or based on specific points.

Point!

! • The equation of a circle with center $(a,\ b)$ and radius r is $(x-a)^2+(y-b)^2=r^2$(Standard form). Especially when the origin is the center, $x^2+y^2=r^2$.

• The equation of a circle can also be expressed as $x^2+y^2+lx+my+n=0$(General form).

The expanded form of the standard form.

! When the center and radius of the circle are known, use $(x-a)^2+(y-b)^2=r^2$

! When $x^2+y^2+lx+my+n=0$ is transformed into $(x-a)^2+(y-b)^2=k$, if $k>0$, it represents a circle with center $(a,\ b)$ and radius $\sqrt{k}$.

Warm Up

Solve the following problems.

1. Find the equation of the circle with center $(3, -2)$ and radius 5

2. Determine what kind of figure $x^2+y^2-6x+2y+6=0$ represents.

Explanation

1 Since the center and radius are known, use $(x-a)^2+(y-b)^2=r^2$.

$(x-3)^2+(y+2)^2=5^2$

$(x-3)^2+(y+2)^2=25$

2 Transform into the form $(x-a)^2+(y-b)^2=k$.

$x^2+y^2-6x+2y+6=0$

$(x-3)^2-9+(y+1)^2-1+6=0$

Simplifying this gives $(x-3)^2+(y+1)^2=2^2$.

Therefore, the required figure is a circle with center $(3,-1)$ and radius 2

If $k>0$, it is a circle with center (a, b) and radius $\sqrt{k}$.

Complete the square for x and y.

When identifying a circle, always state its center and radius.

Try

Solve the following problems.

1 Find the equations of the following circles.

(a) Center $(-3, 1)$, radius 2 **(b)** Center is the origin, radius 4

2 Determine what kind of figure $x^2+y^2-4x-6y+4=0$ represents.

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Solve the following problems.

1 Find the equations of the following circles.

(a) Center $(3, 4)$, radius 6

(b) Center $(-2, 6)$, radius $2\sqrt{3}$

(c) Center is the origin, radius 5

2 Determine what kind of figures the following equations represent.

(a) $x^2+y^2-4x+8y+4=0$ **(b)** $x^2+y^2-4x=0$

(c) $x^2+y^2+2x-4y-20=0$ **(d)** $x^2+y^2+4x-6y=0$

★ Challenge

Find the equation of the circle passing through the two points A(2,5) and B(8,1) such that its center lies on the straight line whose equation is $x+y=8$

4-17
MATHEMATICS

Equation of a Circle (2)

Chapter 4 Geometry and Equations

Point!

! When the center and radius of a circle are known, use

$(x-a)^2+(y-b)^2=r^2$

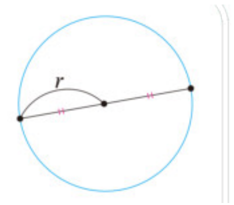
• The radius of a circle can be found by the distance between the **center** and a **point on the circle** .

• The center of a circle where the coordinates of both ends of its diameter are known is the **midpoint of the diameter** .

! When 3 points on the circle are known, use $x^2+y^2+lx+my+n=0$

Learning Outcomes

Derive and use the different forms of the equation of a circle—given its center and radius, or based on specific points.



Warm Up

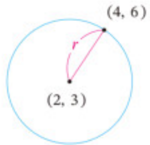
Find the equations of the following circles.

1 A circle with center (2, 3) passing through point (4, 6)

2 A circle with 2 points A(1, 3) and B(3, − 5) as both ends of its diameter

3 A circle passing through 3 points (1, 1), (2, − 1), and (3, 2)

Explanation



1 Since the center of the circle is known, use

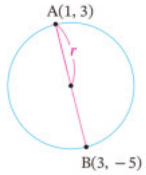
$(x-a)^2+(y-b)^2=r^2$

Find the radius of this circle.

The radius of a circle is the distance between the center and a point on the circle.

$\sqrt{(2-4)^2+(3-6)^2}=\sqrt{13}$

Therefore, the required equation of the circle is $(x-2)^2+(y-3)^2=13$



2 Since the center of the circle with 2 points A(1, 3) and B(3, −5) as both ends of its diameter is the midpoint of AB,

The center is the midpoint of the diameter.

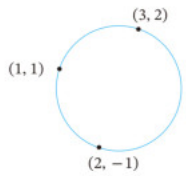
$$\left(\frac{1+3}{2},\ \frac{3+(-5)}{2}\right)=(2,\ -1)$$

Also, since this circle passes through point A, finding the radius gives:

The radius of a circle is the distance between the center and a point on the circle.

$\sqrt{(2-1)^2+(-1-3)^2}=\sqrt{17}$

Therefore, the required equation of the circle is $(x-2)^2+(y+1)^2=17$



3 Since 3 points on a circle are known, use

$x^2 + y^2 + lx + my + n = 0$

Substitute the 3 points on a circle to set up a system of equations.

Since it passes through point $(1, 1)$, $1^2 + 1^2 + l \cdot 1 + m \cdot 1 + n = 0$

Simplifying this gives $l + m + n + 2 = 0$......(i)

Since it passes through point $(2, -1)$, $2^2 + (-1)^2 + l \cdot 2 + m \cdot (-1) + n = 0$

Simplifying this gives $2l - m + n + 5 = 0$ ......(ii)

Since it passes through point $(3, 2)$, $3^2 + 2^2 + l \cdot 3 + m \cdot 2 + n = 0$

Simplifying this gives $3l + 2m + n + 13 = 0$ ......(iii)

Solving (i), (ii), and (iii) simultaneously gives $l = -5$, $m = -1$, and $n = 4$

Therefore, the required equation of the circle is $\underline{x^2 + y^2 - 5x - y + 4 = 0}$

Unless a format is specified, you can leave it in this form.

Try

Find the equations of the following circles.

1. A circle with center $(-3, 1)$ passing through point $(4, -2)$
2. A circle with 2 points $(3, 4)$ and $(5, -2)$ as both ends of its diameter
3. A circle passing through 3 points $(0, 1)$, $(-1, 2)$, and $(3, 1)$

Work with a partner

Collaborate with your group to solve question 3 then discuss the answer.

Exercise

Find the equations of the following circles.

1. A circle with center $(1, -3)$ passing through point $(-3, 0)$
2. A circle with center $(-4, 1)$ passing through point $(3, -2)$
3. A circle with center $(2, -1)$ passing through the origin
4. A circle with the origin as its center passing through point $(4, -3)$
5. A circle with 2 points $(4, 7)$ and $(-2, -1)$ as both ends of its diameter
6. A circle with 2 points $(4, -2)$ and $(-6, -2)$ as both ends of its diameter
7. A circle with 2 points $(-3, 0)$ and $(-1, 2)$ as both ends of its diameter
8. A circle with 2 points $(-3, 6)$ and $(3, -2)$ as both ends of its diameter
9. A circle passing through 3 points $(0, 0)$, $(-1, 1)$, and $(2, 4)$
10. A circle passing through 3 points $(-4, 6)$, $(0, 8)$, and $(3, -1)$
11. A circle passing through 3 points $(1, 1)$, $(5, -1)$, and $(-3, -7)$

★ Challenge

During an electrophysiology study of the heart, cardiologists map the electrical activity of a patient's atrium to locate the origin point of an abnormal heart rhythm (arrhythmia). The wavefront of the electrical anomaly spreads outward in a perfect circle.

A mapping catheter detects the electrical wave passing through three specific sensor coordinates on a diagnostic grid

Sensor 1: (0, 2) Sensor 2: (2, 0) Sensor 3: (4, 6)

Find the mathematical equation of the circle formed by this electrical wave to determine its center point (the arrhythmia focus) and its radius.

4-18

MATHEMATICS

Equation of a Circle (3)

Chapter 4 Geometry and Equations

Point!

! When finding the equation of a circle tangent to an axis, always draw a diagram to visualize.

! The radius of a circle tangent to the x—axis or y—axis can be found by the **distance between the center and the point of tangency** .

Learning Outcomes

Derive and use the different forms of the equation of a circle—given its center and radius, or based on specific points.

Warm Up

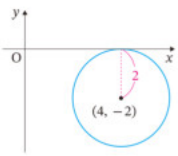
Find the equations of the following circles.

1 A circle with center $(4, -2)$ tangent to the x-axis

2 A circle passing through point $(-2, 1)$ and tangent to both the x-axis and the y—axis

3 A circle whose center lies on the line $y = -x$, passing through 2 points $(5, 2)$ and $(-2, 3)$

Explanation



1 Since the center of the circle is known, use

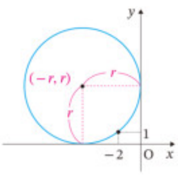
$(x-a)^2+(y-b)^2=r^2$

Find the radius of this circle by drawing a diagram.

From the figure on the left, the radius of this circle is 2

Distance between the center and the point of tangency.

Therefore, the required equation of the circle is $(x-4)^2+(y+2)^2=4$



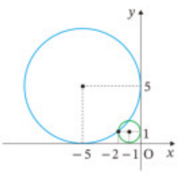
2 Draw a diagram that fits the conditions.

Let the radius of the required circle be $r(r>0)$. From the figure on the left, the coordinates of the center are $(-r, r)$.

Next, to find the radius, create an equation regarding the radius.

Since this circle passes through point $(-2, 1)$:

The radius is the distance between the center and a point on a circle.



$\sqrt{(-2+r)^2+(1-r)^2}=r$

$(-2+r)^2+(1-r)^2=r^2$

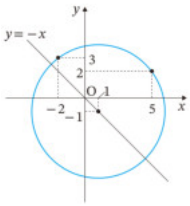
Square both sides.

Simplifying this gives $r^2-6r+5=0$

Solving this gives $r=1,\ 5$

Noting that $r>0$, the required equations of the circles are:

$(x+1)^2+(y-1)^2=1,\ (x+5)^2+(y-5)^2=25$



3 Since the center lies on the line $y = -x$, the coordinates of the center of the required circle can be set as $(a, -a)$.

Let the radius be $r(r > 0)$. Since this circle passes through point $(5, 2)$:

$$\sqrt{(5-a)^2 + (2+a)^2} = r$$

$$(5-a)^2 + (2+a)^2 = r^2 \ldots\ldots \text{(i)}$$

Also, since it passes through point $(-2, 3)$:

$$\sqrt{(-2-a)^2 + (3+a)^2}$$

$$(-2-a)^2 + (3+a)^2 = r^2 \quad \ldots\ldots \text{(ii)}$$

Solving (i) and (ii) simultaneously gives $a = 1,\ r^2 = 25$

Therefore, the required equation of the circle is $\underline{(x-1)^2 + (y+1)^2 = 25}$.

Create an equation regarding the radius.

Since we are finding the equation of a circle, we only need to find r^2.

Try

Find the equations of the following circles.

1 A circle with center $(-2, 3)$ tangent to the y-axis

2 A circle passing through point $(1, 2)$ and tangent to both the x-axis and the y—axis

3 A circle whose center lies on the line $y = x + 5$, passing through the origin and point $(1, 2)$

Work with a partner

Collaborate with your group to solve question 3 then discuss the answer.

Exercise

Find the equations of the following circles.

1 A circle with center $(-1, 4)$ tangent to the x-axis

2 A circle with center $(3, -4)$ tangent to the x-axis

3 A circle with center $(5, -4)$ tangent to the x-axis

4 A circle with center $(-1, 6)$ tangent to the y-axis

5 A circle with center $(2, -5)$ tangent to the y-axis

6 A circle with center $(-3, -4)$ tangent to the y-axis

7 A circle passing through point $(-1, -2)$ and tangent to both the x-axis and the y—axis

8 A circle passing through point $(-2, -1)$ and tangent to both the x-axis and the y—axis

9 A circle passing through point $(8, -1)$ and tangent to both the x-axis and the y—axis

10 A circle whose center lies on the line $y = x + 1$, passing through 2 points $(3, 4)$ and $(1, 4)$

11 A circle whose center lies on the line $y = 2x - 6$, passing through 2 points $(3, 7)$ and $(-1, 3)$

★ Challenge

An arc-based radiotherapy gantry rotates in a circular path. Its center of rotation is constrained to the track along the x-axis. The trajectory must intersect two spatial calibration checkpoints: P1(-2, 2) and P2(1, 1). Find the circle's standard equation, its center of rotation, and the exact beam radius.

4-19

MATHEMATICS

Conditions for Forming a Circle

Chapter 4 Geometry and Equations

Point!

! When the equation $x^2 + y^2 + lx + my + n = 0$ is transformed into the form $(x-a)^2 + (y-b)^2 = $ (constant), if **(constant) > 0** , then this equation represents a circle.

Learning Outcomes

Identify the conditions required for forming the equation of a circle.

Warm Up

Find the range of values for the constant k such that the equation $x^2 + y^2 - 2kx + 2ky + 4k + 6 = 0$ represents a circle.

Explanation

Transforming $x^2 + y^2 - 2kx + 2ky + 4k + 6 = 0$ gives:

$(x-k)^2 - k^2 + (y+k)^2 - k^2 + 4k + 6 = 0$

Transform into the form $(x-a)^2 + (y-b)^2 = $ (constant).

Simplifying this gives:

$(x-k)^2 + (y+k)^2 = 2k^2 - 4k - 6$

If (constant) > 0, it is a circle.

This equation represents a circle when $2k^2 - 4k - 6 > 0$

Solving this gives $\underline{k < -1 \text{ or } k > 3}$

Try

Find the range of values for the constant k such that the equation $x^2 + y^2 + 2kx - 4ky + 4k^2 + 6 = 0$ represents a circle.

Exercise

Solve the following problems.

1 Find the range of values for the constant k such that the equation $x^2 + y^2 + 2x - y + k = 0$ represents a circle.

2 Find the range of values for the constant k such that the equation $x^2 + y^2 + 6kx - 2ky + 28k + 6 = 0$ represents a circle.

3 Find the range of values for the constant k such that the equation $x^2 + y^2 + 2kx - 2(k+3)y + 4k^2 + 1 = 0$ represents a circle.

★ Challenge

An automated ultrasonic welder generates an acoustic stress boundary modeled by the equation:
$x^2 + y^2 + 6x - 2ky + 2k^2 - 4k + 9 = 0$
For a stable, real circular safety perimeter to exist around the machine, the radius of this equation must be a real, positive number. Determine the exact range of values for the power setting constant k that guarantees this equation represents a real circle.

4-20

MATHEMATICS

Points of Intersection of a Circle and a Line (1)

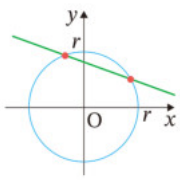
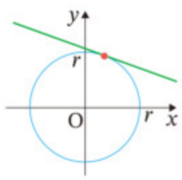
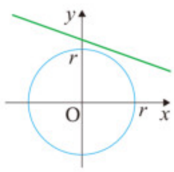
Chapter 4 Geometry and Equations

Point!

! When finding the coordinates or the number of points of intersection between a circle and a line, solve the equations of the circle and the line simultaneously.

• When finding the coordinates, find the solutions to the simultaneous equations.

• When finding the number of points, create a quadratic equation and check the **sign of the discriminant D** .

Sign of D	$D > 0$	$D = 0$	$D < 0$
Number of points of intersection	**2 points**	**1 point**	**0 points**
Position relationship between circle and line	 **Intersects at two distinct points**	 **Touches**	 **Has no points of intersection**

Learning Outcomes

Calculate the points of intersection between a circle and a straight line.

Warm Up

Solve the following problems.

1 Find the coordinates of the points of intersection of the circle $(x+2)^2 + y^2 = 10$ and the line $y = -2x + 1$

2 Find the number of points of intersection of the circle $x^2 + y^2 = 2$ and the line $x - y + 3 = 0$

Explanation

1 Solve the equations of the circle and the line simultaneously.

$$\begin{cases} (x+2)^2 + y^2 = 10 \ldots\ldots \text{(i)} \\ y = -2x + 1 \ \ldots\ldots \text{(ii)} \end{cases}$$

Substituting (ii)into (i) and simplifying gives:

$5x^2 - 5 = 0$

Solving this gives $x = \pm 1$

Substitute into the linear equation.

From (ii), when $x = 1, y = -1$; when $x = -1, y = 3$

Therefore, the coordinates of the required points are $\underline{(1,\ -1),\ (-1,\ 3)}$

2 Solve the equations of the circle and the line simultaneously.

$$\begin{cases} x^2 + y^2 = 2 \ldots\ldots \text{(i)} \\ x - y + 3 = 0 \ \ldots\ldots \text{(ii)} \end{cases}$$

From (ii), $y = x + 3$ ...(ii)'

Substituting (ii)' into (i) and simplifying gives:

$2x^2 + 6x + 7 = 0$

Let the discriminant of this equation be D :

$D = 6^2 - 4 \cdot 2 \cdot 7 = -20 < 0$

Therefore, the number of points of intersection is $\underline{0}$.

Try

Solve the following problems.

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

1 Find the coordinates of the points of intersection of the circle $x^2 + y^2 = 4$ and the line $x + y = 2$

2 Find the number of points of intersection of the circle $x^2 + y^2 = 10$ and the line $y = 2x - 5$

Exercise

Solve the following problems.

1 Find the coordinates of the points of intersection for the following circles and lines.

(a) $x^2 + y^2 = 10, y = 3x - 10$

(b) $x^2 + y^2 = 5,\ 2x - y + 5 = 0$

(c) $(x-1)^2 + y^2 = 10, x + 3y - 1 = 0$

2 Find the number of points of intersection for the following circles and lines.

(a) $x^2 + y^2 = 5, y = x + 2$

(b) $x^2 + y^2 = 2,\ 2x + 3y - 6 = 0$

(c) $(x+1)^2 + (y-2)^2 = 5, x + 2y + 2 = 0$

4-21 MATHEMATICS

Points of Intersection of a Circle and a Line (2)

Chapter 4 Geometry and Equations

Learning Outcomes

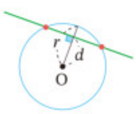
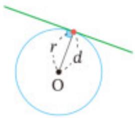
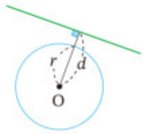
Calculate the points of intersection between a circle and a straight line.

Point!

! How to solve problems where the positional relationship between a circle and a line is a condition:

• If the equation of the line is unknown, use the discriminant based on the positional relationship. Intersects at 2 distinct points ⇒ $D > 0$, Tangent ⇒ $D = 0$, Has no points of intersection ⇒ $D < 0$

• If the radius of the circle is unknown, use the size relationship between (radius r) and (distance d between the center of the circle and the line) based on the positional relationship.

Positional relationship between a circle and a line:	 **Intersects at two distinct points**	 **Touches**	 **Has no points of intersection**
Size relationship between r and d :	$r > d$	$r = d$	$r < d$

Warm Up

Solve the following problems.

1 If the circle $x^2 + y^2 = 5$ and the line $y = 2x - k$ intersect at 2 distinct points, find the range of values for the constant k.

2 If the circle $x^2 + y^2 = r^2$ and the line $2x + y - 5 = 0$ are tangent, find the value of the radius r of the circle.

Explanation

1 Since the equation of the line is unknown, solve the equations of the circle and the line simultaneously, and use the discriminant.

$$5x^2 - 4kx + k^2 - 5 = 0$$

Let the discriminant of this equation be D,

$$D = (-4k)^2 - 4 \cdot 5 \cdot (k^2 - 5)$$

$$= -4k^2 + 100 > 0$$

When they intersect at 2 distinct points, $D > 0$

Solving this gives $-5 < k < 5$

2 Since the radius of the circle is unknown, find the distance d between the center of the circle $(0, 0)$ and the line.

$$d = \frac{|2 \cdot 0 + 1 \cdot 0 - 5|}{\sqrt{2^2 + 1^2}}$$
$$= \frac{|-5|}{\sqrt{5}}$$
$$= \sqrt{5}$$

Therefore $r = \sqrt{5}$

When they are tangent, $r = d$.

Try

Solve the following problems.

1 If the circle $x^2 + y^2 = 4$ and the line $y = mx + 4$ are tangent, find the value of the constant m.

2 If the circle $x^2 + y^2 = r^2$ and the line $x - 3y - 10 = 0$ have no points of intersection, find the range of values for the radius r of the circle.

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Solve the following problems.

1 If the circle $x^2 + y^2 = 8$ and the line $y = x + m$ are tangent, find the value of the constant m.

2 If the circle $x^2 + y^2 = 10$ and the line $y = 3x + n$ have no points of intersection, find the range of values for the constant n.

3 If the circle $x^2 + y^2 = 5$ and the line $y = 2x + m$ have points of intersection, find the range of values for the constant m.

4 If the circle $x^2 + y^2 = r^2$ and the line $y = x + 2$ are tangent, find the value of the radius r of the circle.

5 If the circle $x^2 + y^2 = r^2$ and the line $4x - y + 17 = 0$ intersect at 2 distinct points, find the range of values for the radius r of the circle.

★ Challenge

A coastal radar station maps its coverage area as a circular zone centered at the origin:

$x^2 + y^2 = r^2$

A straight international shipping lane follows the path modeled by the linear equation:

$3x + 4y - 10 = 0$

To comply with maritime privacy laws, the radar's signal must remain completely clear of the shipping lane, meaning they share no points of intersection. Since a radius must be a positive physical distance ($r > 0$), determine the exact range of values for the radius r to maintain this separation.

4-22

MATHEMATICS

Length of a Chord

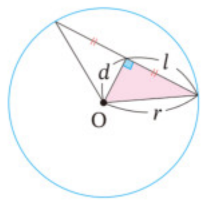
Chapter 4 Geometry and Equations

Point!

! In problems finding the length of a chord, first draw a diagram and use **Pythagoras' theorem** . Let r denote the radius of the circle, d the length of the perpendicular drawn from the centre of the circle to the chord, and l half the length of the chord. Since the perpendicular drawn from the centre of the circle to the chord bisects the chord, we have $r^2 = d^2 + l^2$.

Learning Outcomes

Compute the length of the common chord formed by the intersection of a circle and a line.



Warm Up

Solve the following problems.

1 Find the length of the chord cut off by the circle $x^2 + y^2 = 4$ on the line $x + y - 1 = 0$

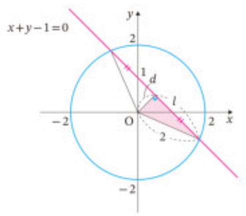
2 Find the value of the constant k such that the length of the line segment connecting the 2 points of intersection of the circle $x^2 + y^2 = 3$ and the line $x - y + k = 0$ is 2

Explanation

1 Since it is a problem to find the length of a chord, first draw a diagram to visualize.

Let d be the length of the perpendicular dropped from the center of the circle $(0, 0)$ to the line $x + y - 1 = 0$:

The length of the perpendicular is the distance between the point and the line.



$$d = \frac{|1 \cdot 0 + 1 \cdot 0 - 1|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

Let half the length of the required chord be l. Since the radius of the circle is 2, from Pythagoras' theorem:

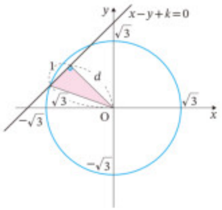
$l^2 = 2^2 - d^2$

$$l^2 = 4 - \left(\frac{1}{\sqrt{2}}\right)^2$$

$l^2 = \frac{7}{2}$ From $l > 0, l = \frac{\sqrt{14}}{2}$

Therefore, the required length of the chord is $\underline{2l = \sqrt{14}}$.

2 Since the line segment connecting the 2 points of intersection of the circle $x^2 + y^2 = 3$ and the line $x - y + k = 0$ forms a chord, draw a diagram to visualize. Let d be the length of the perpendicular dropped from the center of the circle $(0, 0)$ to the line $x - y + k = 0$:



$$d = \frac{|\,1 \cdot 0 + (-1) \cdot 0 + k\,|}{\sqrt{1^2 + (-1)^2}}$$

$$d = \frac{|\,k\,|}{\sqrt{2}} \text{......}(i)$$

Also, since half the length of the chord is 1 and the radius of the circle is $\sqrt{3}$, from Pythagoras' theorem:

$$d^2 = (\sqrt{3})^2 - 1^2$$

$d^2 = 2$ Since $d > 0, d = \sqrt{2}$......(ii)

From (i) and (ii), $\dfrac{|\,k\,|}{\sqrt{2}} = \sqrt{2}$

$|k| = 2$

Therefore, $\underline{k = \pm 2}$

When removing the absolute value sign, add ±.

Try

Solve the following problems.

1 Find the length of the chord cut off by the circle $(x-2)^2 + (y+1)^2 = 9$ on the line $x + 2y - 5 = 0$

2 Find the value of the constant k such that the length of the line segment connecting the 2 points of intersection of the circle $x^2 + y^2 = 16$ and the line $y = x + k$ is $\sqrt{2}$

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Solve the following problems.

1 Find the length of the chord cut off by the circle $x^2 + y^2 = 10$ on the line $2x - y = 5$

2 Find the length of the chord cut off by the circle $(x-2)^2 + (y-1)^2 = 5$ on the line $x + y = 4$

3 Find the length of the chord cut off by the circle $x^2 + y^2 = 3$ on the line $x - y = 2$

4 Find the value of the constant k such that the length of the line segment connecting the 2 points of intersection of the circle $x^2 + y^2 = 10$ and the line $y = x + k$ is $4\sqrt{2}$

5 Find the value of the constant k such that the length of the line segment connecting the 2 points of intersection of the circle $(x-1)^2 + y^2 = 5$ and the line $y = x + k$ is $3\sqrt{2}$

★ Challenge

Find the value of the constant k such that the length of the line segment connecting the 2 points of intersection of the circle $x^2 + y^2 + 4x - 2y - 7 = 0$ and the line $y = x + k$ is 4

4-23

MATHEMATICS

Tangent to a Circle

Chapter 4 Geometry and Equations

Point!

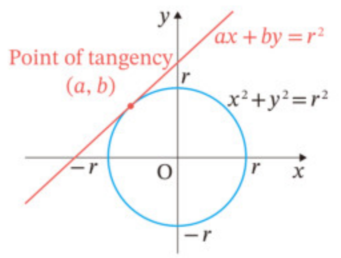
! The equation of the tangent to the circle $x^2+y^2=r^2$ at point $(a,\ b)$ on it is: $\boldsymbol{ax+by=r^2}$

! For the equation of a tangent drawn from a point not on the circle, consider the coordinates of the **point of tangency as (a, b)**.

Learning Outcomes

Derive the equation of a tangent to a circle.

When the problem states "drawn from ~"



Warm Up

Find the equations of the following tangents.

1. Tangent to the circle $x^2+y^2=25$ at point $(4, 3)$
2. Tangent drawn from point P$(1, 3)$ to the circle $x^2+y^2=5$

Explanation

1. Since the point of tangency is $(4, 3)$,

   $4x+3y=25$

When the problem states "at a given point"

2. Let the coordinates of the point of tangency be $(a,\ b)$.

   The equation of the tangent is $ax+by=5$

   Since this passes through point P$(1, 3)$, $a+3b=5$ ......(i)

   Also, since the point of tangency lies on $x^2+y^2=5$, $a^2+b^2=5$ ......(ii)

   Solve (i) and (ii) simultaneously.

   From (i), $a=-3b+5$...... (i)'

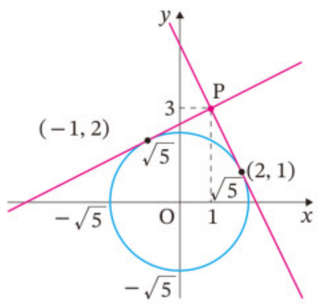
   Substituting (i)' into (ii) gives $(-3b+5)^2+b^2=5$

   Solving this gives $b=1,\ 2$

   Substituting into (i)', when $b=1$, $a=2$; when $b=2$, $a=-1$

   Therefore, since the point of tangency is $(2, 1)$ or $(-1, 2)$,

   the required equations of the tangents are $2x+y=5$ and $-x+2y=5$



There are 2 tangents drawn from a point outside the circle.

When the problem states "at ~"

Try

Find the equations of the following tangents.

1. Tangent to the circle $x^2 + y^2 = 34$ at point $(5, 3)$

2. Tangent drawn from point $(-2, 4)$ to the circle $x^2 + y^2 = 10$

Work with a partner

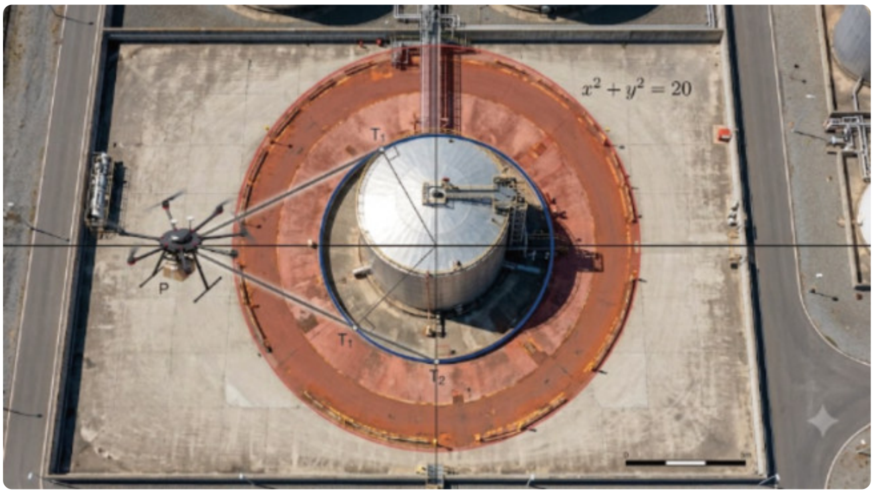
Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Find the equations of the following tangents.

1. Tangent to the circle $x^2 + y^2 = 10$ at point $(3, -1)$
2. Tangent to the circle $x^2 + y^2 = 29$ at point $(-5, -2)$
3. Tangent to the circle $x^2 + y^2 = 5$ at point $(-2, 1)$
4. Tangent drawn from point $(-1, 3)$ to the circle $x^2 + y^2 = 5$
5. Tangent drawn from point $(2, 1)$ to the circle $x^2 + y^2 = 1$
6. Tangent drawn from point $(2, 6)$ to the circle $x^2 + y^2 = 20$

★ Challenge



An autonomous delivery drone is navigating an industrial yard. The yard contains a circular high-voltage storage tank centered at the origin, with its hazard zone modeled by the equation:

$x^2 + y^2 = 20$

The drone is currently hovering at a staging point mapped at the coordinates P(2, 6). To safely inspect the perimeter without entering the hazard zone, the drone must align its optical sensors along a path that is perfectly tangent to the circle from its current position.

Find the equations of the two available tangent sightlines the drone can utilize, and pinpoint the exact coordinates where these lines touch the boundary of the hazard zone.

4-24

MATHEMATICS

Positional Relationship Between Two Circles

Chapter 4 Geometry and Equations

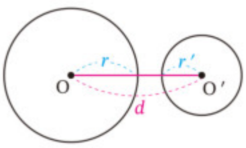
Point!

! Positional relationship between two circles (Review of Math A)

Let the radius of circle O be r, the radius of circle O′ be r' (where $r > r'$), and the distance between their centers be d. The positional relationship between the two circles and the relationship between r, r', and d are as follows:

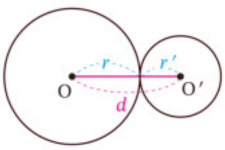
Are external to each other

$r + r' < d$



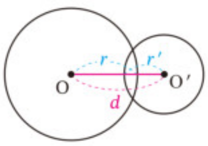
Touch externally

$r + r' = d$



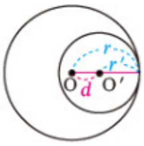
Intersect at 2 points

$r - r' < d < r + r'$



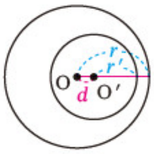
Touch internally

$r - r' = d$



One is inside the other

$r - r' > d$



Learning Outcomes

Determine the relative position between two circles (separate, tangent, or intersecting).

Warm Up

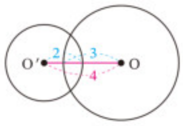
Solve the following problems.

1 State the positional relationship between the circle $x^2 + y^2 = 4$ and the circle $(x - 4)^2 + y^2 = 9$

2 Find the equation of the circle with center $(3, 1)$ that touches the circle $(x - 1)^2 + (y + 3)^2 = 5$ internally.

Explanation

1 Since the centers of the 2 circles are (0, 0) and (4, 0),



$d = \sqrt{(4-0)^2 + (0-0)^2} = 4$

Since the radii of the two circles are 2 and 3 respectively,

From $3 - 2 < 4 < 3 + 2$,

$r - r' < d < r + r'$

Therefore, the 2 circles <u>intersect at 2 points</u>.

2 Since the center of the circle $(x-1)^2 + (y+3)^2 = 5$ is $(1, -3)$,

$d = \sqrt{(3-1)^2 + \{1-(-3)\}^2} = 2\sqrt{5}$

Since the 2 circles touch internally, $r - r' = d$.

The radius of the circle $(x-1)^2 + (y+3)^2 = 5$ is $\sqrt{5}$, and

since $\sqrt{5} < d, r' = \sqrt{5}$

Therefore, $r - \sqrt{5} = 2\sqrt{5}$

$r = 3\sqrt{5}$

Therefore, the required equation of the circle is:

$\underline{(x-3)^2 + (y-1)^2 = 45}$

Try

Solve the following problems.

1 Choose the positional relationship of the following 2 circles from **A** to **E**, and answer with the letter.

(a) Circle $x^2 + y^2 = 9$ and circle $(x-3)^2 + (y-4)^2 = 4$

(b) Circle $(x-1)^2 + y^2 = 52$ and circle $x^2 + y^2 + 2x - 6y = 3$

A. Are external to each other

B. Touch externally .

C. Touch internally .

D. One is inside the other

E. Intersect at 2 points

2 Find the equation of the circle with center $(-3, -9)$ that touches the circle $(x+6)^2 + (y+5)^2 = 1$ externally.

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Solve the following problems.

1 Choose the positional relationship of the following 2 circles from **A** to **E**, and answer with the letter.

(a) Circle $x^2 + y^2 = 4$ and circle $(x-1)^2 + y^2 = 1$

(b) Circle $x^2 + y^2 = 18$ and circle $(x+2)^2 + (y-2)^2 = 2$

(c) Circle $(x+3)^2 + y^2 = 16$ and circle $x^2 + y^2 - 2x - 8y - 8 = 0$

(d) Circle $x^2 + (y-3)^2 = 45$ and circle $x^2 + y^2 + 2x - 2y = 3$

A. Are external to each other	**B**. Touch externally	**C**. Touch internally
D. One is inside the other	**E**. Intersect at 2 points	

2 Find the equation of the circle with center $(3, -9)$ that touches the circle $(x-3)^2 + y^2 = 9$ internally.

★ Challenge

Find the equation of the circle with center $(-1, -2)$ that touches the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ externally.

4-25

MATHEMATICS

Points of Intersection of Two Circles

Chapter 4 Geometry and Equations

Learning Outcomes

Find the coordinates of the points of intersection of two circles.

Point!

To find the coordinates of the points of intersection of two circles or the equation of the line passing through the points of intersection, create and solve simultaneous equations.

Warm Up

Solve the following problems.

1 Find the equation of the line passing through the points of intersection of the circles $x^2 + y^2 = 4$ and

$x^2 + y^2 + 2x - 4y = 9$

2 Find the coordinates of the points of intersection of the circles $x^2 + y^2 = 10$ and $x^2 + y^2 + 2x - y = 5$

Explanation

1 Solving the equations of the two circles simultaneously:

$$\begin{cases} x^2 + y^2 = 4 \text{......(i)} \\ x^2 + y^2 + 2x - 4y = 9 \text{ ......(ii)} \end{cases}$$

Subtracting (ii) from (i) gives:

$-2x + 4y = -5$

$y = \frac{1}{2}x - \frac{5}{4}$

Therefore, the required equation is:

$y = \frac{1}{2}x - \frac{5}{4}$

2 Solving the equations of the two circles simultaneously:

$$\begin{cases} x^2 + y^2 = 10 \text{......(i)} \\ x^2 + y^2 + 2x - y = 5 \text{ ......(ii)} \end{cases}$$

Subtracting (ii) from (i) gives:

$-2x + y = 5$

$y = 2x + 5$ ......(iii)

Substituting (iii) into (i) gives:

$x^2 + (2x + 5)^2 = 10$

$5x^2 + 20x + 15 = 0$

$x^2 + 4x + 3 = 0$

$x = -3, \ -1$

Substituting these values into (iii):

When $x = -3, y = -1$

When $x = -1, y = 3$

Therefore, the coordinates of the points of intersection are $(-3, -1)$, $(-1, 3)$.

Try

Solve the following problems.

1 Find the equation of the line passing through the points of intersection of the circles $x^2 + y^2 = 6$ and $x^2 + y^2 + 3x + 3y = 9$

2 Find the coordinates of the points of intersection of the circles $x^2 + y^2 = 4$ and $(x - 4)^2 + (y - 2)^2 = 16$

Work with a partner

Collaborate with your group to solve question 2 then discuss the answer.

Exercise

Solve the following problems.

1 Find the equation of the line passing through the points of intersection of the following 2 circles.

(a) The circles $x^2 + y^2 = 10$ and $x^2 + y^2 - 12x + 6y = 3$

(b) The circles $x^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 4$

2 Find the coordinates of the points of intersection of the circles $x^2 + y^2 = 5$ and $(x - 2)^2 + (y - 2)^2 = 1$

★ Challenge

Two omnidirectional radar stations are tracking a drone.
Station Alpha is covering the range of $x^2 + y^2 = 25$
Station Beta is covering the range of $(x + 5)^2 + (y + 5)^2 = 65$
To pinpoint the exact locations where the drone transitions between both radar perimeters, find the intersection points of their signal boundaries.

4-26 MATHEMATICS

Circle Passing Through the Points of Intersection of Two Circles

Chapter 4 Geometry and Equations

Learning Outcomes

Find the equation of the circle that passes through the points of intersection of two other circles.

Point!

A curve passing through the points of intersection of the two circles $x^2+y^2+px+qy+r=0$ and $x^2+y^2+lx+my+n=0$ can be expressed as follows: $\boldsymbol{k(x^2+y^2+px+qy+r)+(x^2+y^2+lx+my+n)=0}$

When writing the final equation of a circle, ensure the coefficients of x^2 and y^2 equal to 1

Multiply one equation by k and add it to the other equation.

Warm Up

Find the equation of the circle passing through the points of intersection of two circles $x^2+y^2-2x-6y+5=0$ and $x^2+y^2=5$, and point $(3, 0)$.

Explanation

The figure passing through the points of intersection of these 2 circles can be expressed as $k(x^2+y^2-2x-6y+5)+(x^2+y^2-5)=0$ ......(i)

Since the circle passes through the point $(3, 0)$:

$$k(3^2+0^2-2\cdot 3-6\cdot 0+5)+3^2+0^2-5=0$$

$$8k+4=0$$

Therefore, $k=-\dfrac{1}{2}$

Substituting this into (i) and calculating gives:

$$\frac{1}{2}x^2+\frac{1}{2}y^2+x+3y-\frac{15}{2}=0$$

Make the coefficients of x^2 and y^2 equal to 1

Therefore, $x^2+y^2+2x+6y-15=0$

Try

Find the equation of the circle passing through the points of intersection of two circles $x^2+y^2-4x-2y-8=0$ and $x^2+y^2=4$, and the origin.

Work with a partner

Collaborate with your group to solve the try section then discuss the answer.

Exercise

Solve the following problems.

1. Find the equation of the circle passing through the points of intersection of 2 circles $x^2+y^2-2x-4y+3=0$ and $x^2+y^2=4$, and point $(3, 0)$.

2. Find the equation of the circle passing through the points of intersection of 2 circles $x^2+y^2-6x-8y=0$ and $x^2+y^2-1=0$, and point $(-3, 2)$.

3. Find the equation of the circle passing through the points of intersection of 2 circles $x^2+y^2-2x-2y=0$ and $x^2+y^2-6x-4y+12=0$, and point $(4, 0)$.

★ Challenge

Find the equation of the circle passing through the points of intersection of 2 circles $x^2+y^2-2x-6y+6=0$ and $x^2+y^2+4x-2y-11=0$, and point $(2, 2)$.

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